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GEOMETRICAL

PROBLEMS.

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GEOMETRICAL PROBLEMS

DEDUCIBLE

FROM THE FIRST SIX BOOKS OF EUCLID,

ARRANGED AND SOLVED:

TO WHICH IS ADDED

AN APPENDIX

CONTAINING THE ELEMENTS OF

Plane Trigonometry.

FOR THE USE OF THE YOUNGER STUDENTS.

By MILES BLAND, B.D.

FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

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The following pages contain a collection of Problems, which are for the most part an easy application of the Elements of Euclid. They are arranged in what seemed to be the most natural order: The 1st section comprises such as contain the properties of straight lines and angles; the 2nd straight lines and circles: the 3rd straight lines and triangles; and the 4th parallelograms, squares and polygons. The 5th section contains those which require lines to be drawn in certain directions, but which involve properties of rectangles or squares, or such others as were excluded from the three first. The 6th comprises those by which figures are described, and also inscribed in or circumscribed about each other. The 7th comprehends such as contain the properties of triangles described in or about circles; the 8th those which contain the squares or rectangles of lines connected with circles; and the 9th the construction of triangles. To these is added an Appendix, intended to contain so much of the Elements of Plane Trigonometry, as is necessary for understanding those parts of Natural Philosophy which are the common subjects of Lectures in the University. The Reader who wishes for farther information, is referred to Mr. Woodhouse's treatise, or that of Cagnoli, to the latter of which are appended extensive Tables of trigonometrical formulæ.

From this performance the only credit expected is that of having endeavoured to place principles in a clear light, and to render a service to the younger students by setting before them a series of Problems, on the solution of which they are recommended to exercise their own ingenuity; for which purpose a table of Contents has been prefixed.

To the Syndics of the University Press, who from the funds which are placed at their disposal, with great readiness agreed to bear a considerable part of the expence of printing, most sincere thanks are due; and this opportunity is taken of gratefully acknowledging the obligation.

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SECTION I. Page 1.

1. From a given point to draw the shortest line possible to a given straight line.

2. If a perpendicular be drawn bisecting a given straight line, any point in this perpendicular is at equal distances, and any point without the perpendicular is at unequal distances, from the extremities of the line.

3. Through a given point, to draw a straight line which shall make equal angles with two straight lines given in position.

4. From two given points, to draw two equal straight lines, which shall meet in the same point of a line given in position.

5. From two given points on the same side of a line given in position, to draw two lines which shall meet in that line, and make equal angles with it.

6. From two given points on the same side of a line given in position, to draw two lines which shall meet in a point in this line, so that their sum shall be less than the sum of any two lines drawn from the same points and terminated at any other point in the same line.

7. Of all straight lines which can be drawn from a given point to an indefinite straight line, that which is nearer to the perpendicular is less than the more remote. And from the same point there cannot be drawn more than two straight lines equal to each other, viz. one on each side of the perpendicular.

8. Through a given point, to draw a straight line so that the parts of it intercepted between that point and perpendiculars drawn from two other given points may have a given ratio.

9. From a given point between two indefinite straight lines given in position, to draw a line which shall be terminated by the given lines, and bisected in the given point.

10. From a given point without two indefinite straight lines given in position, to draw a line such that the parts intercepted by the point and the lines may have a given ratio.

11. From a given point to draw a straight line which shall cut off from lines containing a given angle, segments that shall have a given ratio.

12. If from a given point any number of straight lines be drawn to a straight line given in position; to determine the locus of the points of section, which divide them in a given ratio.

13. A straight line being drawn parallel to one of the lines containing a given angle, and produced to meet the other; through a given point within the angle to draw a line cutting the other three, so that the part intercepted between the two parallel lines may have a given ratio to the part intercepted between the given point and the other line.

14. Two parallel lines being given in position; to draw a third such, that if from any point in it lines be drawn at given angles to the parallel lines, the intercepted parts may have a given ratio.

15. If three straight lines drawn from the same point and in the same direction be in continued proportion, and from that point also a line equal to the mean proportional be inclined at any angle; the lines joining the extremity of this line and of the proportionals will contain equal angles.

16. To trisect a right angle.

17. To trisect a given finite straight line.

18. To divide a given straight line into any number of equal parts.

Cor. To divide a straight line into any number of parts having a given ratio.

19. To divide a given finite straight line harmonically.

20. If a given finite straight line be harmonically divided, and from its extremities and the points of division lines be drawn to meet

in any point, so that those from the extremities of the second proportional may be perpendicular to each other; the line drawn from the extremity of this proportional will bisect the angle formed by the lines drawn from the extremities of the other two.

21. If a straight line be drawn through any point in the line bisecting a given angle, and produced to cut the sides containing that angle, as also a line drawn from the angle perpendicular to the bisecting line; it will be harmonically divided.

22. If from a given point there be drawn three straight lines forming angles less than right angles, and from another given point without them a line be drawn intersecting the others, so as to be harmonically divided; then will all lines drawn from that point meeting the three lines be harmonically divided.

23. If a straight line be divided into two equal and also into two unequal parts, and be produced, so that the part produced may have to the whole line so produced, the same ratio that the unequal segments of the line have to each other; then shall the distances of the point of unequal section from one extremity of the given line, from its middle point, from the extremity of the part produced, and from the other extremity of the given line, be proportionals.

24. Three points being given; to determine another, through which if any straight line be drawn, perpendiculars upon it from two of the former shall together be equal to the perpendicular from the third.

25. From a given point in one of two straight lines given in position, to draw a line to cut the other, so that if from the point of intersection a perpendicular be let fall upon the former, the segment intercepted between it and the given point together with the first drawn line may be equal to a given line.

26. One of the lines which contain a given angle is also given. To determine a point in it such that if from thence to the indefinite line there be drawn a line having a given ratio to that segment of it which is adjacent to the given angle; the line so drawn and the other segment of the given line may together be equal to another given line.

27. Two straight lines and a point in each are given in position; to determine the position of another point in each, so that the straight

line joining these latter points may be equal to a given line, and their respective distances from the former points in a given ratio.

28. If a straight line be divided into any two parts and produced, so that the segments may have the same ratio that the whole line produced has to the part produced, and from the extremities of the given line perpendiculars be erected; then any line drawn through the point of section, meeting these perpendiculars, will be divided at that point into parts which have the same ratio, that those lines have which are drawn from the extremity of the produced line to the points of intersection with the perpendicular.

29. From two given points to draw two straight lines which shall contain a given angle, and meet two lines given in position, so that the parts intercepted between those points and the lines may have a given ratio.

30. The length of one of two lines which contain a given angle being given; to draw, from a given point without them, a straight line which shall cut the given line produced, so that the part produced may be in a given ratio to the part cut off from the indefinite line.

31. From two given straight lines to cut off two parts which may have a given ratio; so that the ratio of the remaining parts may also be equal to the ratio of two other given lines.

32. Three lines being given in position; to determine a point in one of them, from which if two lines be drawn at given angles to the other two, the two lines so drawn may together be equal to a given line.

33. If from a given point two straight lines be drawn including a given angle and having a given ratio, and one of them be always terminated by a straight line given in position; to determine the locus of the extremity of the other.

34. If from two given points straight lines be drawn containing a given angle, and from each of them segments be cut off having a given ratio; and the extremities of the segments of the lines drawn from one of the points be in a straight line given in position; to determine the locus of the extremities of the segments of lines drawn from the other.

SECTION II. Page 24.

1. If a straight line be drawn to touch a circle, and be parallel to a chord; the point of contact will be the middle point of the arc cut off by that chord.

Cor. 1. Parallel lines placed in a circle cut off equal parts of the circumference.

Cor. 2. The two straight lines in a circle which join the extremities of two parallel chords are equal to each other.

2. If from a point without a circle two straight lines be drawn to the concave part of the circumference, making equal angles with the line joining the same point and the centre, the parts of these lines which are intercepted within the circle are equal.

3. Of all straight lines which can be drawn from two given points to meet on the convex circumference of a given circle; the sum of those two will be the least, which make equal angles with the tangent at the point of concourse.

4. If a circle be described on the radius of another circle; any straight line drawn from the point where they meet to the outer circumference, is bisected by the interior one.

5. If two circles cut each other, and from either point of intersection diameters be drawn; the extremities of these diameters and the other point of intersection shall be in the same straight line.

6. If two circles cut each other, the straight line joining their two points of intersection is bisected at right angles by the straight line joining their centres.

7. To draw a straight line which shall touch two given circles.

8. If a line touching two circles cut another line joining their centres, the segments of the latter will be to each other, as the diameters of the circles.

9. If a straight line touch the interior of two concentric circles, and be placed in the outer; it will be bisected at the point of contact.

10. If any number of equal straight lines be placed in a circle; to determine the locus of their points of bisection.

11. If from a point in the circumference of a circle any number of chords be drawn; the locus of their points of bisection will be a circle.

12. If on the radius of a given semicircle, another semicircle be described, and from the extremity of the diameters any lines be drawn cutting the circumferences, and produced, so that the part produced may always have a given ratio to the part intercepted between the two circumferences; to determine the locus of the extremities of these lines.

13. If from a given point without a given circle straight lines be drawn and terminated by the circumference; to determine the locus of the points which divide them in a given ratio.

14. Having given the radius of a circle; to determine its centre when the circle touches two given lines which are not parallel.

15. Through three given points which are not in the same straight line, a circle may be described; but no other circle can pass through the same points.

16. From two given points on the same side of a line given in position, to draw two straight lines which shall contain a given angle, and be terminated in that line.

17. If from the extremities of any chord in a circle perpendiculars be drawn, meeting a diameter; the points of intersection are equally distant from the centre.

18. If from the extremities of the diameter of a semicircle perpendiculars be let fall on any line cutting the semicircle; the parts intercepted between those perpendiculars and the circumference are equal.

19. In a given circle to place a straight line parallel to a given straight line, and having a given ratio to it.

20. Through a given point, either within or without a given circle, to draw a straight line, the part of which intercepted by the circle shall be equal to a given line, not greater than the diameter of the circle.

21. From a given point in the diameter of a semicircle produced, to draw a line cutting the semicircle, so that lines drawn from the

points of intersection to the extremities of the diameter, cutting each other, may have a given ratio.

22. From the circumference of a given circle, to draw to a straight line given in position, a line which shall be equal and parallel to a given straight line.

23. The bases of two given circular segments being in the same straight line; to determine a point in it such, that a line being drawn through it making a given angle, the part intercepted between the circumferences of the circles may be equal to a given line.

24. If two chords of a given circle intersect each other, the angle of their inclination is equal to half the angle at the centre which stands on an arc equal to the sum or difference of the arcs intercepted between them, according as they meet within or without the circle.

25. If from a point without two circles which do not meet each other, two lines be drawn to their centres, which have the same ratio that their radii have; the angle contained by tangents drawn from that point towards the same parts will be equal to the angle contained by lines drawn to the centres.

26. To determine the Arithmetic, Geometric and Harmonic means between two given straight lines.

27. If on each side of any point in a circle any number of equal arcs be taken, and the extremities of each pair joined: the sum of the chords so drawn will be equal to the last chord produced to meet a line drawn from the given point through the extremity of the first arc.

28. If the circumference of a semicircle be divided into an odd number of equal parts, and through the points which are equally distant from the diameter lines be drawn; the segments of these lines intercepted between radii drawn to the extremities of the most remote, will together be equal to a radius of the circle.

29. If from the extremities and the point of bisection of any arc of a circle, lines be drawn to any point in the opposite circumference; the sum of those drawn from the extremities will have to that from the point of bisection, the same ratio that the line joining the extremities has to that joining one of them and the point of bisection.

30. If two equal circles cut each other, and from either point of intersection a circle be described cutting them; the points where this circle cuts them, and the other point of intersection of the equal circles are in the same straight line.

31. If two equal circles cut each other, and from either point of intersection a line be drawn meeting the circumferences; the part of it intercepted between the circumferences will be bisected by the circle whose diameter is the common chord of the equal circles.

32. If two circles touch each other externally or internally; any straight line drawn through the point of contact will cut off similar segments.

33. If two circles touch each other externally or internally; two straight lines drawn through the point of contact will intercept arcs, the chords of which are parallel.

34. If two circles touch each other externally or internally; any two straight lines drawn through the point of contact, and terminated both ways by the circumference, will be cut proportionally by the circumference.

35. If two circles touch each other externally, and parallel diameters be drawn; the straight line joining the extremities of these diameters will pass through the point of contact.

36. If two circles touch each other and also touch a straight line; the part of the line between the points of contact is a mean proportional between the diameters of the circles.

37. If two circles touch each other externally, and the line joining their centres be produced to the circumferences; and from its middle point as a centre with any radius whatever a circle be described, and any line placed in it passing through the point of contact; the parts of the line intercepted between the circumference of this circle and each of the others will be equal.

38. If from the point of contact of two circles which touch each other internally, any number of lines be drawn; and through the points, where these intersect the circumferences, lines be drawn from any other point in each circumference, and produced to meet; the angles formed by these lines will be equal.

39. If two circles touch each other internally, and any two per-

pendiculars to their common diameter be produced to cut the circumferences; the lines joining the points of intersection and the point of contact are proportional.

40. If three circles, whose diameters are in continued proportion, touch each other internally, and from the extremity of the least diameter passing through the point of contact a perpendicular be drawn, meeting the circumferences of the other two circles; this diameter and the lines joining the points of intersection and contact are in continued proportion.

41. If a common tangent be drawn to any number of circles which touch each other internally, and from any point in this tangent as a centre, a circle be described cutting the others, and from this centre lines be drawn through the intersections of the circles respectively; the segments of them within each circle will be equal.

42. If from any point in the diameter of a circle produced, a tangent be drawn; a perpendicular from the point of contact to the diameter will divide it into segments which have the same ratio that the distances of the point without the circle from each extremity of the diameter, have to each other.

43. If from the extremity of the diameter of a given semicircle a straight line be drawn in it, equal to the radius, and from the centre a perpendicular let fall upon it and produced to the circumference; it will be a mean proportional between the lines drawn from the point of intersection with the circumference to the extremities of the diameter.

44. If from the extremity of the diameter of a circle, two lines be drawn, one of which cuts a perpendicular to the diameter, and the other is drawn to the point where the perpendicular meets the circumference; the latter of these lines is a mean proportional between the cutting line, and that part of it which is intercepted between the perpendicular and the extremity of the diameter.

45. In the diameter of a circle produced, to determine a point, from which a tangent drawn to the circumference, shall be equal to the diameter.

46. To determine a point in the perpendicular at the extremity of the diameter of a semicircle, from which if a line be drawn to the

other extremity of the diameter, the part without the circle may be equal to a given straight line.

47. Through a given point without a given circle, to draw a straight line to cut the circle, so that the two perpendiculars drawn from the points of intersection to that diameter which passes through the given point, may together be equal to a given line, not greater than the diameter of the circle.

48. If from each extremity of any number of equal adjacent arcs in the circumference of a circle, lines be drawn through two given points in the opposite circumference, and produced till they meet; the angles formed by these lines will be equal.

49. To determine a point in the circumference of a circle, from which lines drawn to two other given points, shall have a given ratio.

50. If any point be taken in the diameter of a circle, which is not the centre; of all the chords which can be drawn through that point, that is the least which is at right angles to the diameter.

51. If from any point without a circle lines be drawn touching it; the angle contained by the tangents is double the angle contained by the line joining the points of contact and the diameter drawn through one of them.

52. If from the extremities of the diameter of a circle tangents be drawn, and produced to intersect a tangent to any point of the circumference; the straight lines joining the points of intersection and the centre of the circle form a right angle.

53. If from the extremities of the diameter of a circle tangents be drawn; any other tangent to the circle, terminated by them, is so divided at the point of contact, that the radius of the circle is a mean proportional between its segments.

54. Two circles being given in magnitude and position; to find a point in the circumference of one of them, to which if a tangent be drawn cutting the circumference of the other, the part of it intercepted between the two circumferences may be equal to a given line.

55. To draw a straight line cutting two concentric circles, so that the part of it which is intercepted by the circumference of the greater may be double the part intercepted by the circumference of the less.

56. If two circles intersect each other, the centre of the one being in the circumference of the other, and any line be drawn from that centre; the parts of it which are cut off by the common chord and the two circumferences will be in continued proportion.

57. If a semicircle be described on the side of a quadrant, and from any point in the quadrantal arc a radius be drawn, the part of this radius intercepted between the quadrant and semicircle, is equal to the perpendicular let fall from the same point on their common tangent.

Cor. Any chord of the semicircle drawn from the centre of the quadrant is equal to the perpendicular drawn to the other side, from the point in which the chord produced meets the quadrantal arc.

58. If a semicircle be described on the side of a quadrant, and a line be drawn from the centre of the quadrant to a common tangent; this line, the parts of it cut off by the circumferences of the quadrant, and of the semicircle, and the segment of the diameter of the semicircle made by a perpendicular from the point where the line meets its circumference, are in continued proportion.

59. If the chord of a quadrant be made the diameter of a semicircle, and from its extremities two straight lines be drawn to any point in the circumference of the semicircle; the segment of the greater line intercepted between the two circumferences shall be equal to the less of the two lines.

60. If two circles cut each other, so that the circumference of one passes through the centre of the other, and from either point of intersection a straight line be drawn cutting both circumferences; the part intercepted between the two circumferences will be equal to the chord drawn from the other point of intersection to the point where it meets the inner circumference.

61. If from each extremity of the diameter of a circle lines be drawn to any two points in the circumference; the sums of the lines so drawn to each point will have to one another the same ratio that the lines have, which join those points and the opposite extremity of a diameter perpendicular to the former.

62. If from any two points in the circumference of a circle there be drawn two straight lines to a point in a tangent to that circle; they will make the greatest angle when drawn to the point of contact.

63. From a given point within a given circle to draw a straight line which shall make with the circumference an angle less than the angle made by any other line drawn from that point.

64. To determine a point in the arc of a quadrant, from which if lines be drawn to the centre and the point of bisection of the radius, they shall contain the greatest possible angle.

65. If the radius of a circle be a mean proportional to two distances from the centre in the same straight line; the lines drawn from their extremities to any point in the circumference will have the same ratio that the distances of these points from the circumference have.

66. Two circles being given in position and magnitude; to draw a straight line cutting them so that the chords in each circle may be equal to a given line, not greater than the diameter of the smaller circle.

67. To determine a point in the arc of a quadrant, through which if a tangent be drawn meeting the sides of the quadrant produced, the intercepted parts may have a given ratio.

68. If a tangent be drawn to a circle at the extremity of a chord which cuts the diameter at right angles, and from any point in it a perpendicular be let fall; the segment of the diameter intercepted between that perpendicular and chord is to the intercepted part of the tangent as the chord is to the diameter.

69. If a straight line be placed in a circle, and from its extremities perpendiculars be let fall upon any diameter; these perpendiculars together will have to the part of the diameter intercepted between them, the same ratio that a line placed in the circle perpendicular to the former line, has to the former line itself.

70. In a circle to place a straight line of given length, so that perpendiculars drawn to it from two given points in the circumference may have a given ratio.

71. If from any point in the arc of a segment of a circle a line be drawn perpendicular to the base; and from the greater segment of the base and arc, parts be cut off respectively equal to the less; the remaining part of the base shall be equal to the chord of the remaining arc.

72. If from the point of bisection of any arc of a circle a perpendicular be drawn to the diameter which passes through one extremity; it will bisect the segment of the chord cut off by the line joining the point of bisection of the arc and the other extremity of the diameter.

73. In a given circle to draw a chord parallel to a straight line given in position; so that the chord and perpendicular drawn to it from the centre may together be equal to a given line.

74. Through a given point within a given circle, to draw a straight line such that the parts of it intercepted between that point and the circumference may have a given ratio.

75. From two given points in the circumference of a given circle, to draw two lines to a point in the circumference, which shall cut a line given in position, so that the part of it intercepted by them may be equal to a given line.

76. If a chord and diameter of a circle intersect each other at any angle, and a perpendicular to the chord be drawn from either extremity of it, meeting the circumference and diameter produced; the whole perpendicular has to the part of it without the circle, the same ratio that the greater segment of the chord has to the less.

77. If from the extremities of any chord of a circle, perpendiculars to it be drawn and produced to cut a diameter; and from the points of intersection with the diameter lines be drawn to a point in the chord so as to make equal angles with it; these lines together will be equal to the diameter of the circle.

78. If from a point without a circle two straight lines be drawn, one of which touches and the other cuts the circle; a line drawn from the same point in any direction, equal to the tangent, will be parallel to the chord of the arc intercepted by two lines drawn from its other extremity to the former intersections of the circle.

79. If from a point without a circle two straight lines be drawn touching it, and from one point of contact a perpendicular be drawn to that diameter which passes through the other; this perpendicular will be bisected by the line joining the point without the circle and the other extremity of the diameter.

80. If any chord in a circle be bisected by another, and produced to meet the tangents drawn from the extremities of the bisecting line; the parts intercepted between the tangents and the circumferences are equal.

81. If one chord in a circle bisect another, and tangents drawn from the extremities of each be produced to meet; the line joining their points of intersection will be parallel to the bisected chord.

82. If from a point without a circle two lines be drawn touching the circle, and from the extremities of any diameter lines be drawn to the points of contact, cutting each other within the circle; the line produced, which joins their intersection and the point without the circle, will be perpendicular to the diameter.

83. If on opposite sides of the same extremity of the diameter of a circle equal arcs be taken, and from the extremities of these arcs lines be drawn to any point in the circumference, one of which cuts the diameter, and the other the diameter produced; the distances of the points of intersection from the extremities of the diameter are proportional to each other.

84. If from the extremities of any chord in a circle, perpendiculars be drawn to a diameter, and from either extremity of that diameter a perpendicular be drawn to the chord; it will divide it into segments, which are respectively mean proportionals between the segments of the diameter cut off by the perpendiculars.

85. If from any point in the diameter of a semicircle, a perpendicular be drawn, meeting the circumference, and on it as a diameter a circle be described, to the centre of which a line is drawn from the farther extremity of the diameter of the semicircle, cutting its circumference; and through the point of intersection another line be drawn from the extremity of the perpendicular, meeting the diameter of the semicircle; this diameter will be divided into three segments which are in continued proportion.

86. If from a point without a given circle, any two lines be drawn cutting the circle; to determine a point in the circumference, such that the sum of the perpendiculars from it upon these lines may be equal to a given line.

87. If two circles cut each other, and any two points be taken

in the circumference of one of them, through which lines are drawn from the points of intersection and produced to the circumference of the other; the straight lines joining the extremities of those which are drawn through the same point, are equal.

88. If two circles cut each other; the greatest line that can be drawn through the point of intersection is that which is parallel to the line joining their centres.

89. Having given the radii of two circles which cut each other, and the distance of their centres; to draw a straight line of given length through their point of intersection, so as to terminate in their circumferences.

90. If two circles cut each other; to draw from one of the points of intersection a straight line meeting the circles, so that the part of it intercepted between the circumferences may be equal to a given line.

91. If two circles cut each other; to draw from the point of intersection two lines, the parts of which intercepted between the circumferences may have a given ratio.

92. If a semicircle be described on the common chord of two intersecting circles, and a line drawn from one extremity of the chord, cutting the two circles; the part intercepted between the two shall be divided by the semicircle into segments proportional to perpendiculars drawn in those circles from the other extremity of the chord.

93. Two circles being given, the circumference of one of which passes through the centre of the other; to draw a chord from that centre, such that a perpendicular let fall upon it from a given point, may bisect that part of it which is intercepted between the circumferences.

94. If any number of circles cut each other in the same points, and from one of these points any number of lines be drawn; the parts of them which are intercepted between the several circumferences have the same ratio.

95. In a given circle to place a straight line cutting two radii which are perpendicular to each other, in such a manner that the line itself may be trisected.

96. If a straight line be divided into any two parts, and upon the whole line and one of the parts, as diameters, semicircles be described; to determine a point in the less diameter, from which if a perpendicular be drawn cutting the circumferences, and the points of intersection and the extremities of the respective diameters be joined, and these lines produced to meet; the parts of them without the semicircles may have a given ratio.

97. If a straight line be divided into any two parts, and from the point of section a perpendicular be erected, which is a mean proportional between one of the parts and the whole line, and a circle described through the extremities of the line and the perpendicular; the whole line, the perpendicular, the aforesaid part, and a perpendicular drawn from its extremity to the circumference will be in continued proportion.

98. If the tangents drawn to every two of three unequal circles be produced till they meet, the points of intersection will be in a straight line.

99. If from the extremities of the diameter of a circle any number of chords be drawn, two and two intersecting each other in a perpendicular to that diameter; the lines joining the extremities of every corresponding two will meet the diameter produced in the same point.

100. If from a given point in the diameter of a semicircle produced, three straight lines be drawn, one of which is inclined at a given angle to the diameter, another touches the semicircle, and the third cuts it, in such a manner, that the distance of the given point from the nearer extremity of the diameter, and the perpendiculars drawn from that extremity on the three aforesaid lines may be proportional; then will the lines, which join the extremities of the diameter and of that part of the cutting line which is within the circle, intersect each other in an angle equal to the given angle.

SECTION III. Page 92.

1. Any side of a triangle is greater than the difference between the other two sides.

2. In any right-angled triangle, the straight line joining the right angle and the bisection of the hypothenuse is equal to half the hypothenuse.

3. If from any point within an equilateral triangle perpendiculars be drawn to the sides; they are together equal to a perpendicular drawn from any of the angles to the opposite side.

4. If the points of bisection of the sides of a given triangle be joined; the triangle so formed will be one fourth of the given triangle.

5. The difference of the angles at the base of any triangle is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.

6. If from one of the equal angles of an isosceles triangle any line be drawn to the opposite side, and from the same point a line be drawn to the opposite side produced, so that the part intercepted between them may be equal to the former; the angle contained by the side of the triangle and the first drawn line is double the angle contained by the base and the latter.

7. If from the extremity of the base of an isosceles triangle, a line equal to one of the sides be drawn to meet the opposite side; the angle formed by this line and the base produced is equal to three times either of the equal angles of the triangle.

8. The sum of the sides of an isosceles triangle is less than the sum of the sides of any other triangle on the same base and between the same parallels.

9. If from one of the equal angles of an isosceles triangle, a perpendicular be drawn to the opposite side; the part of it intercepted by a perpendicular from the vertex will have to one of the equal sides, the same ratio that the segment of the base has to the perpendicular upon the base.

10. If from any point in the base of an isosceles triangle lines be

drawn to the opposite sides, making equal angles with the base; the triangles formed by these lines, the segments of the base, and the lines joining the intersections of the sides and the angles opposite, will be equal.

11. If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides; these together shall be equal to a perpendicular drawn from either extremity of the base to the opposite side.

12. Of all triangles having the same vertical angle, and whose bases pass through a given point, the least is that whose base is bisected in the given point.

13. If from the angles at the base of a triangle, perpendiculars be let fall on a line which bisects the vertical angle; the part of this line intercepted between these perpendiculars will be bisected by a perpendicular from the middle of the base.

14. If from one of the angles at the base of a triangle a line be drawn parallel to the opposite side and from any point in it lines be drawn making any angles with the sides (produced, if necessary); they will have the same ratio that lines have which are drawn parallel to them from the other angles, and terminated by the same sides.

15. To bisect a given triangle by a line draw from one of its angles.

16. To bisect a given triangle by a line drawn from a given point in one of its sides.

17. To determine a point within a given triangle, from which lines drawn to the several angles, will divide the triangle into three equal parts.

18. To trisect a given triangle from a given point within it.

19. From a given point in the side of a triangle, to draw lines which will divide the triangle into any number of parts which shall have a given ratio.

20. If two exterior angles of a triangle be bisected, and from the point of intersection of the bisecting lines, a line be drawn to the opposite angle of the triangle; it will bisect that angle.

21. If in two triangles, the vertical angle of the one be equal to that of the other, and one other angle of the former be equal to the exterior angle at the base of the latter; the sides about the third angle of the former shall be proportional to those about the interior angle at the base of the latter.

22. In a given triangle to draw a line parallel to one of the sides, so that it may be a mean proportional between the segments of the base.

23. To draw a line parallel to the common base of two triangles which have different altitudes, so that the parts of it intercepted by the sides may have a given ratio.

24. If the base of a triangle be produced so that the whole may be to the part produced in the duplicate ratio of the sides; the line joining the vertex, and the extremity of the part produced will be a mean proportional between the whole line produced and the part produced.

25. To determine a point within a given triangle, which will divide a line parallel to the base into two segments, such that the excess of each segment above the perpendicular distance between the parallel lines may be to each other in the duplicate ratio of the respective segments.

26. If perpendiculars be drawn to two sides of a triangle from any two points therein; the distance of their concourse from that of the two sides will be to the distance between the two points as either side is to the perpendicular drawn from its extremity upon the other.

27. If the three sides of a triangle be bisected, the perpendiculars drawn to the sides at the three points of bisection, will meet in the same point.

28. If from the three angles of a triangle lines be drawn to the points of bisection of the opposite sides, these lines intersect each other in the same point.

29. The three straight lines which bisect the three angles of a triangle, meet in the same point.

30. If the three angles of a triangle be bisected, and one of the bisecting lines be produced to the opposite side; the angle contained

by this line produced, and one of the others is equal to the angle contained by the third and a perpendicular drawn from the common point of intersection of the three lines, to the aforesaid side.

31. In a right-angled triangle, if a straight line be drawn parallel to the hypothenuse, and cutting the perpendicular drawn from the right angle; and through the point of intersection a line be drawn from one of the acute angles to the opposite side, and the extremity of this line and of the perpendicular be joined; the locus of its intersection with the line parallel to the hypothenuse will be a straight line.

32. If from the angles of a triangle, lines, each equal to a given line, be drawn to the opposite sides (produced if necessary); and from any point within, lines be drawn parallel to these, and meeting the sides of the triangle; these lines will together be equal to the given line.

33. If the sides of a triangle be cut proportionally, and lines be drawn from the points of section to the opposite angles; the intersections of these lines will be in the same line, viz. that drawn from the vertex to the middle of the base.

34. If from any point in one side of a triangle, two lines be drawn, one to the opposite angle, and the other parallel to the base, and the former intersect a line drawn from the vertex bisecting the base; this point of intersection, that of the line parallel to the base and the third side, and the third angular point are in the same straight line.

35. If one side of a triangle be divided into any two parts, and from the point of section two straight lines be drawn parallel to, and terminating at the other sides, and the points of termination be joined; and any other line be drawn parallel to either of the two former lines, so as to intersect the other, and to terminate in the sides of the triangle; then the two extreme parts of the three segments into which the line so drawn is divided, will always be in the ratio of the segments of the first divided line.

36. If through the point of bisection of the base of a triangle, any line be drawn intersecting one side of the triangle and the other produced, and meeting a parallel to the base from the vertex; this line will be cut harmonically.

37. If from either angle of a triangle, a line be drawn intersecting that which joins the vertex and the bisection of the base, the opposite side, and the line from the vertex parallel to the base; it will be cut harmonically.

38. To draw a line from one of the angles at the base of a triangle, so that the part of it cut off by a line drawn from the vertex parallel to the base, may have a given ratio to the part cut off by the opposite side.

39. To determine that point in the base produced of a right-angled triangle, from which the line drawn to the angle opposite to the base shall have the same ratio to the base produced, which the perpendicular has to the base itself.

40. If the base of any triangle be divided into two parts by a line which is a mean proportional between them, and which being drawn parallel to the second side is terminated in the third; any line parallel to the base will be divided by the mean proportional (produced if necessary) into segments which will be to each other inversely as the whole mean proportional to that segment which is terminated in the third side of the triangle.

41. If from the extremities of the base of any triangle, two straight lines be drawn intersecting each other in the perpendicular, and terminating in the opposite sides; straight lines drawn from thence to the intersection of the perpendicular with the base, will make equal angles with the base.

42. In any triangle, the intersection of the perpendiculars drawn from the angles to the opposite sides, the intersection of the lines from the angles to the middle of the opposite sides, and the intersection of the perpendiculars from the middle of the sides, are all in the same straight line. And the distances of those points from one another are in a given ratio.

43. If straight lines be drawn from the angles of a triangle through any point, either within or without the triangle, to meet the sides, and the lines joining these points of intersection and the sides of the triangle be produced to meet; the three points of concourse will be in the same straight line.

SECTION IV. Page 120.

1. The diameters of a rhombus bisect each other at right angles.

2. If the opposite sides or opposite angles of a quadrilateral figure be equal; the figure will be a parallelogram.

3. To bisect a parallelogram by a line drawn from a point in one of its sides.

4. If from any point in the diameter (or diameter produced) of a parallelogram straight lines be drawn to the opposite angles; they will cut off equal triangles.

5. From one of the angles of a parallelogram to draw a line to the opposite side, which shall be equal to that side together with the segment of it which is intercepted between the line and the opposite angle.

6. If from one of the angles of a parallelogram a straight line be drawn, cutting the diameter, a side and a side produced; the segment intercepted between the angle and the diameter is a mean proportional between the segments intercepted between the diameter and the sides.

7. The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of two opposite sides, are together half of the parallelogram.

8. If a straight line be drawn parallel to one of the sides of a parallelogram, and one extremity of this line be joined to the opposite one of the parallel side, by a line which also cuts the diameter; the segments of the diameter made by this line will be reciprocally proportional to the segments of that part of it which is intercepted between the side and the parallel line.

9. If two lines be drawn parallel and equal to the adjacent sides of a parallelogram; the lines joining their extremities, if produced, will meet the diameter in the same point.

10. If in the sides of a square, at equal distances from the four angles, four other points be taken, one in each side; the figure contained by the straight lines which join them shall also be a square.

11. The sum of the diagonals of a trapezium is less than the sum of any four lines which can be drawn to the four angles from any point within the figure, except from the intersection of the diagonals.

12. Every trapezium is divided by its diagonals into four triangles proportional to each other.

13. If two opposite angles of a trapezium be right angles; the angles subtended by either side at the two opposite angular points shall be equal.

14. To determine the figure formed by joining the points of bisection of the sides of a trapezium; and its ratio to the trapezium.

15. To determine the figure formed by joining the points where the diagonals of the trapezium cut the parallelogram (in the last problem); and its ratio to the trapezium.

16. If two sides of a trapezium be parallel; its area is equal to half that of a parallelogram whose base is the sum of those two sides, and altitude the perpendicular distance between them.

17. If from any angle of a rectangular parallelogram a line be drawn to the opposite side, and from the adjacent angle of the trapezium thus formed another be drawn perpendicular to the former; the rectangle contained by these two lines is equal to the given parallelogram.

18. To divide a parallelogram into two parts which shall have a given ratio, by a line drawn parallel to a given line.

19. To bisect a trapezium by a line drawn from one of its angles.

20. To bisect a trapezium by a line drawn from a given point in one of its sides.

21. If two sides of a trapezium be parallel; the triangle contained by either of the other sides, and the two straight lines drawn from its extremities to the bisection of the opposite side, is half the trapezium.

22. To divide a given trapezium, whose opposite sides are parallel, in a given ratio, by a line drawn through a given point, and terminated by the two parallel sides.

23. If a trapezium, which has two of its adjacent angles right angles, be bisected by a line drawn from the middle of one of those

sides which are not parallel; the sum of the parallel sides will have to one of them the same ratio, that the side which is not bisected has to that segment of it which is adjacent to the other.

24. If the sides of an equilateral and equiangular pentagon be produced to meet; the angles formed by these lines are together equal to two right angles.

25. If the sides of an equilateral and equiangular hexagon be produced to meet; the angles formed by these lines are together equal to four right angles.

26. The area of any two parallelograms described on the two sides of a triangle is equal to that of a parallelogram on the base, whose side is equal and parallel to the line drawn from the vertex of the triangle to the intersection of the two sides of the former parallelograms produced to meet.

27. The perimeter of an isosceles triangle is greater than the perimeter of a rectangular parallelogram, which is of the same altitude with, and equal to the given triangle.

28. If from one of the acute angles of a right-angled triangle, a line be drawn to the opposite side; the squares of that side and the line so drawn are together equal to the squares of the segment adjacent to the right angle and of the hypothenuse.

29. In any triangle, if a line be drawn from the vertex at right angles to the base; the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.

30. In any triangle, if a line be drawn from the vertex bisecting the base; the sum of the squares of the two sides of the triangle is double the sum of the squares of the bisecting line and of half the base.

31. If from the three angles of a triangle lines be drawn to the points of bisection of the opposite sides; the squares of the distances between the angles and the common intersection are together one third of the squares of the sides of the triangle.

32. If from any point within or without any rectilineal figure, perpendiculars be let fall on every side; the sum of the squares of the alternate segments made by them will be equal.

33. If from any point within a rectangular parallelogram lines be drawn to the angular points; the sums of the squares of those which are drawn to the opposite angles are equal.

34. The squares of the diagonals of a parallelogram are together equal to the squares of the four sides.

35. If two sides of a trapezium be parallel to each other; the squares of its diagonals are together equal to the squares of its two sides which are not parallel, and twice the rectangle contained by its parallel sides.

36. The squares of the diagonals of a trapezium are together double the squares of the two lines joining the bisections of the opposite sides.

37. The squares of the diagonals of a trapezium are together less than the squares of the four sides, by four times the square of the line joining the points of bisection of the diagonals.

38. In any trapezium, if two opposite sides be bisected; the sum of the squares of the two other sides, together with the squares of the diagonals, is equal to the sum of the squares of the bisected sides, together with four times the square of the line joining those points of bisection.

39. If squares be described on the sides of a right-angled triangle; each of the lines joining the acute angles and the opposite angle of the square, will cut off from the triangle an obtuse-angled triangle, which will be equal to that cut off from the square by a line drawn from the intersection with the side to that angle of the square which is opposite to it.

40. If squares be described on the two sides of a right-angled triangle; the lines joining each of the acute angles of the triangle and the opposite angle of the square will meet the perpendicular drawn from the right angle upon the hypothenuse, in the same point.

41. If squares be described on the three sides of a right-angled triangle, and the extremities of the adjacent sides be joined; the triangles so formed are equal to the given triangle and to each other.

42. If the sides of the square described on the hypothenuse of a right-angled triangle be produced to meet the sides (produced if

necessary) of the squares described upon the legs; they will cut off triangles equiangular and equal to the given triangle.

43. If from the angular points of the squares described upon the sides of a right-angled triangle perpendiculars be let fall upon the hypothenuse produced; they will cut off equal segments; and the perpendiculars will together be equal to the hypothenuse.

44. If on the two sides of a right-angled triangle squares be described; the lines joining the acute angles of the triangle and the opposite angles of the squares will cut off equal segments from the sides: and each of these equal segments will be a mean proportional between the remaining segments.

45. If squares be described on the hypothenuse and sides of a right-angled triangle, and the extremities of the sides of the former and the adjacent sides of the others be joined; the sum of the squares of the lines joining them will be equal to five times the square of the hypothenuse.

46. If a line be drawn parallel to the base of a triangle, and terminated in the sides; to draw a line cutting it, and terminated also by the sides, so that the rectangle contained by their segments may be equal.

47. If the sides, or sides produced, of a triangle be cut by any line; the solids formed by the segments which have not a common extremity are equal.

48. If through any point within a triangle, three lines be drawn parallel to the sides; the solids formed by the alternate segments of these lines are equal.

49. If through any point within a triangle lines be drawn from the angles to cut the opposite sides; the segments of any one side will be to each other in the ratio compounded of the ratios of the segments of the other sides.

50. If from each of the angles of any triangle, a line be drawn through any point within the triangle to the opposite side; the solid formed by the segments thereof, intercepted between the angles and the point, will have to the solid formed by the three remaining segments, the same ratio that the solid formed by the three sides of the triangle has to either of the (equal) solids formed by the alternate segments of the sides.

SECTION V. Page 153.

1. A STRAIGHT line of given length being drawn from the centre at right angles to the plane of a circle; to determine that point in it which is equally distant from the upper end of the line, and the circumference of the circle.

2. To determine a point in a line given in position, to which lines drawn from two given points may have the greatest difference possible.

3. A straight line being divided in two given points; to determine a third such that its distances from the extremities may be proportional to its distances from the given points.

4. In a straight line given in position, to determine a point, at which two straight lines drawn from given points on the same side, will contain the greatest angle.

5. To determine the position of a point, at which lines drawn from three given points shall make with each other angles equal to given angles.

6. To divide a straight line into two parts such that the rectangle contained by them may be equal to the square of their difference.

7. If a straight line be divided into any two parts; to produce it so that the rectangle contained by the whole line so produced and the part produced, may be equal to the rectangle contained by the given line and one segment.

COR. 1. To produce the line, so that the rectangle contained by the whole line and the part produced may be equal to the rectangle contained by two given lines.

COR. 2. To produce the line, so that the rectangle contained by the whole line produced and the part produced may be equal to a given square.

8. To determine two lines such that the sum of their squares may be equal to a given square, and their rectangle equal to a given rectangle.

9. To divide a straight line into two parts, so that the rectangle

contained by the whole and one of the parts may be equal to the square of a given line, which is less than the line to be divided.

10. To divide a given line into two such parts, that the rectangle contained by the whole line, and one of the parts may be (m) times the square of the other part; (m) being whole or fractional.

11. To divide a given line into two such parts, that the square of the one shall be equal to the rectangle contained by the other and a given line.

12. A straight line being given in magnitude and position; to draw to it, from a given point, two lines, whose rectangle shall be equal to a given rectangle, and which shall cut off equal segments from the given line.

13. To draw a straight line which shall touch a given circle, and make with a given line, an angle equal to a given angle.

14. Through a given point to draw a line terminating in two lines given in position, so that the rectangle contained by the two parts may be equal to a given rectangle.

15. From a given point to draw a line cutting two given parallel lines, so that the difference of its segments may be equal to a given line.

16. From a given point without a circle, to draw a straight line cutting the circle, so that the rectangle contained by the part of it without, and the part within the circle shall be equal to a given square.

17. From a given point in the circumference of a semicircle, to draw a straight line meeting the diameter, so that the difference between the squares of this line and a perpendicular to the diameter from the point of intersection may be equal to a given rectangle.

18. From a given point to draw two lines to a third given in position, so that the rectangle contained by those lines may be equal to a given rectangle, and the difference of the angles which they make with that part of the third which is intercepted between them may be equal to a given angle.

19. Two points being given without a given circle; to determine a point in the circumference, from which lines drawn to the two given points shall contain the greatest possible angle.

20. From the bisection of a given arc of a circle, to draw a straight line such that the part of it intercepted between the chord of that and the opposite circumference shall be equal to a given straight line.

21. To draw a straight line through a given point, so that the sum of the perpendiculars to it from two other given points may be equal to a given line.

22. To draw a straight line through one of three points given in position, so that the rectangle contained by the perpendiculars let fall upon it from the other two, may be equal to a given square.

23. A given straight line being divided into two parts; to cut off a part which shall be a mean proportional between the two remaining segments.

24. To draw a straight line making a given angle with one of the sides of a given triangle, so that the triangle cut off may be to the whole in a given ratio.

25. Between two given straight lines containing a given angle, to place a straight line of given length, and subtending that angle, so that the segment of the one of them adjacent to the angle may be to the segment of the other which is not adjacent, in the ratio of two given lines.

26. From two given points to draw two lines to a point in a third, such that the difference of their squares may be equal to a given square.

27. To divide a given straight line into two such parts, that the square of one may be to the excess of a given rectangle above the square of the other, in a given ratio.

28. From any angle of a triangle, not isosceles about the angle, to draw a line without the triangle to the opposite side produced, which shall be a mean proportional between the segments of the side.

29. From the obtuse angle of any triangle, to draw a line within the triangle to the opposite side, which shall be a mean proportional between the segments of the side.

30. From the common extremity of the diameters of two semicircles, given in magnitude and position; to draw a line meeting the

circumferences so that the rectangle contained by the two chords may be equal to a given square.

31. To draw a line parallel to a given line, which shall be terminated by two others given in position, so as to form with them a triangle equal to a given rectilineal figure.

32. To bisect a triangle by a line drawn parallel to one of its sides.

33. To divide a given triangle into any number of parts, having a given ratio to each other, by lines drawn parallel to one of the sides of the triangle.

34. To divide a given triangle into any number of equal parts, by lines drawn parallel to a given line.

35. To divide a trapezium, which has two sides parallel, into any number of equal parts, by lines drawn parallel to those sides.

36. From one of the angular points of a given square, to draw a line meeting one of the opposite sides and the other produced, in such a manner, that the exterior triangle formed thereby may have a given ratio to the square.

37. From a given point in the side produced of a given rectangular parallelogram, to draw a line which shall cut the perpendicular sides and the other side produced, so that the trapezium cut off, which stands on the aforesaid side, may be to the triangle which stands upon the produced part of the opposite side, in a given ratio.

38. Through a given point between two straight lines containing a given angle, to draw a line which shall cut off a triangle equal to a given figure.

39. Between two lines given in position, to draw a line equal to a given line, so that the triangle thus formed may be equal to a given rectilineal figure.

40. From two given lines to cut off two others, so that the remainder of one may have to the part cut off from the other, a given ratio; and the difference of the squares of the other remainder and part cut off from the first may be equal to a given square.

41. From two given lines to cut off two others which shall have a given ratio, so that the difference of the squares of the remainders may be equal to a given square.

42. From two given lines to cut off two others, so that the remainders may have a given ratio, and the sum of the squares of the parts cut off may be equal to the square of a given line.

43. Two points being given in a given straight line; to determine a third, such that the rectangles contained by its distances from each extremity and the given point adjacent to that extremity may be equal.

44. Through the point of intersection of two given circles, to draw a line in such a manner, that the sum of the respective rectangles contained by the parts thereof, which are intercepted between the said point and their circumferences, and given lines *A* and *B*, may be equal to a given square.

45. Through a given point, to draw an indefinite line such, that if lines be drawn from two other given points and forming given angles with it, the rectangle contained by the segments intercepted between the given point and the two lines so drawn, shall be equal to the square of a given line.

46. Through a given point between two straight lines containing a given angle, to draw a line such that a perpendicular upon it from the given angle may have a given ratio to a line drawn from one extremity of it, parallel to a line given in position.

47. Through a given point between two indefinite straight lines, not parallel to one another, to draw a line, which shall be terminated by them, so that the rectangle contained by its segments shall be less than the rectangle contained by the segments of any other line drawn through the same point.

SECTION VI. Page 184.

1. To describe an isosceles triangle on a given finite straight line.

2. To describe a square which shall be equal to the difference of two squares, whose sides are given.

Cor. Hence a mean proportional between the sum and difference of two given lines may be determined.

3. To describe a rectangular parallelogram, which shall be equal to a given square, and have its adjacent sides together equal to a given line.

4. To describe a rectangular parallelogram, which shall be equal to a given square, and have the difference of its adjacent sides equal to a given line.

5. To describe a triangle, which shall be equal to a given equilateral and equiangular pentagon, and of the same altitude.

6. To describe an equilateral triangle, equal to a given isosceles triangle.

7. To describe a parallelogram, the area and perimeter of which shall be respectively equal to the area and perimeter of a given triangle.

8. To describe a parallelogram, which shall be of given altitude, and equiangular and equal to a given parallelogram.

9. To describe a square which shall be equal to the sum of any number of given squares.

10. Having given the difference between the diameter and side of a square; to describe the square.

11. To divide a circle into any number of concentric equal annuli.

COR. To divide it into annuli which shall have a given ratio.

12. In any quadrilateral figure circumscribing a circle, the opposite sides are equal to half the perimeter.

13. If the opposite angles of a quadrilateral figure be equal to two right angles, a circle may be described about it.

14. A quadrilateral figure may have a circle described about it, if the rectangles contained by the segments of the diagonals be equal.

15. If from any point within a regular figure circumscribed about a circle, perpendiculars be drawn to the sides; they will together be equal to that multiple of the semi-diameter which is expressed by the number of the sides of the figure.

16. If the radius of a circle be cut in extreme and mean ratio; the greater segment will be equal to the side of an equilateral and equiangular decagon inscribed in that circle.

17. Any segment of a circle being described on the base of a triangle; to describe on the other sides segments similar to that on the base.

18. If an equilateral triangle be inscribed in a circle; the square described on a side thereof is equal to three times the square described upon the radius.

19. To inscribe a square in a given right-angled isosceles triangle.

20. To inscribe a square in a given quadrant of a circle.

21. To inscribe a square in a given semicircle.

22. To inscribe a square in a given segment of a circle.

23. Having given the distance of the centres of two equal circles which cut each other; to inscribe a square in the space included between the two circumferences.

24. In a given segment of a circle to inscribe a rectangular parallelogram, whose sides shall have a given ratio.

25. In a given circle to inscribe a rectangular parallelogram equal to a given rectilineal figure.

26. In a given segment of a circle to inscribe an isosceles triangle, such that its vertex may be in the middle of the chord, and the base and perpendicular together equal to a given line.

27. In a given triangle, to inscribe a parallelogram similar to a given parallelogram.

28. In a given triangle, to inscribe a triangle similar to a given triangle.

29. In a given equilateral and equiangular pentagon, to inscribe a square.

30. In a given triangle, to inscribe a rhombus, one of whose angles shall be in a given point in the side of the triangle.

31. To inscribe a circle in a given quadrant.

32. To describe a circle, the circumference of which shall pass through a given point, and touch a given straight line in a given point.

33. To describe a circle, which shall pass through a given point, have a given radius, and touch a given straight line.

✓ 34. To describe a circle, which shall pass through two given points, and touch a given straight line.

35. To describe a circle, the circumference of which shall pass through a given point, and touch a circle in a given point; the two points not being in a tangent to the given circle.

36. To describe a circle, the centre of which may be in the perpendicular of a given right-angled triangle, and the circumference pass through the right angle and touch the hypothenuse.

37. To describe a circle, which shall pass through the extremities of a given line, so that if from any point in its circumference a line be drawn making a given angle with the given line; the rectangle contained by the segment it cuts off and the given line, may be equal to the square of the line drawn from the same point to the farther extremity of the given line.

38. To determine a point in the perpendicular let fall from the vertical angle of any triangle on the base; about which as a centre a circle may be described touching the longer side, and passing through the opposite angular point.

39. To describe a circle which shall have a given radius, its centre in a given straight line, and shall also touch another given straight line inclined at a given angle to the former.

40. To describe a circle which shall touch a straight line in a given point, and also touch a given circle.

41. To describe two circles, each having a given radius, which shall touch each other, and the same given straight line on the same side of it.

✓ 42. To describe a circle passing through two given points, and touching a given circle.

✓ 43. To describe a circle which shall pass through a given point, and touch a given circle and a given straight line.

✓ 44. To describe a circle which shall touch a straight line and two circles given in magnitude and position.

45. To describe a circle which shall touch two given straight lines, and pass through a given point between them.

46. To describe a circle which shall touch two given straight lines, and also touch a given circle.

47. To describe a circle which shall touch a circle and straight line, both given in position, and have its centre also in a given straight line.

48. Through two given points within a given circle, to describe a circle, which shall bisect the circumference of the other.

49. Through two given points without a given circle, to describe a circle which shall cut off from the one, an arc equal to a given arc.

50. To describe three circles of equal diameters, which shall touch each other.

51. Every thing remaining as in the last proposition; to describe a circle which shall touch the three circles.

52. To determine how many equal circles may be described round another circle of the same diameter, touching each other and the interior circle.

53. From a given rectangular parallelogram to cut off a gnomon, whose breadth shall be every where the same, and whose area shall be to that of the parallelogram in any given ratio.

54. To describe a triangle equal to a given rectilinear figure, having its vertex in a given point in a side of the figure, and its base in the base (produced if necessary) of the figure.

55. On the base of a given triangle to describe a quadrilateral figure equal to the triangle, and having two of its sides parallel, one of them being the base of the triangle; and one of its angles being an angle at the base, and the other equal to a given angle.

56. A trapezium being given, two of whose sides are parallel; to describe on one of those sides another trapezium, having its opposite side also parallel to this, one of the angles at the base the same as the former, and the other equal to a given angle.

57. If with any point in the circumference of a circle as centre, and distance from its centre as radius, a circular arc be described; and any two chords be drawn, one from the centre of the circular arc, and the other through the point where this cuts the arc, and parallel to the line joining the centres; the segments of each chord intercepted between the circumferences which are concave to each other, will be equal respectively to those of the other between the other circumferences.

58. If the diagonals of an equilateral and equiangular pentagon be drawn to cut one another, the greater segments will be equal to the side of the pentagon; and the diagonals cut each other in extreme and mean ratio.

59. If the sides of a triangle inscribed in the segment of a circle be produced to meet lines drawn from the extremities of the base, forming with it angles equal to the angle in the segment; the rectangle contained by these lines will be equal to the square described on the base.

60. If two triangles (one of them right-angled) have the same base and altitude, and the hypothenuse intersect a line which is drawn bisecting the right angle; a line passing through this point of intersection parallel to the base, and terminated by the sides of the other triangle, shall be a side of the square inscribed within it.

61. If on the side of a rectangular parallelogram, as a diameter, a semicircle be described, and from any point in the circumference lines be drawn through its extremities to meet the opposite side produced; the altitude of the parallelogram will be a mean proportional between the segments cut off.

62. If on the diameter of a semicircle a rectangular parallelogram be described, whose altitude is equal to the chord of half the semicircle, and lines drawn from any point in the circumference to the extremities of the base intersect the diameter, the squares of the distances of each point of section from the farthest extremity of the diameter will be together equal to the square of the diameter.

Cor. The square of the part of the diameter intercepted between the two lines drawn from the point in the semicircle is double of the rectangle contained by the two extreme segments.

63. If on the radius drawn from the point of contact of a circle and its circumscribed square, another circle be described; and from any point in the outer circumference a line be drawn through its centre to the inner circumference, and through the same point another line be drawn parallel to the common tangent to the circles, and terminated by the side of the square and its diagonal; these two lines are equal.

64. If two sides of a trapezium inscribed in a circle be produced, and from the same point in one side produced, a line be

drawn parallel to the other, intersecting the adjacent side of the trapezium, and a second line to the extremity of that other intersecting the circumference: the line joining the two points of intersection will pass through the same point.

65. If the diagonals of a quadrilateral figure inscribed in a circle, cut each other at right angles; the rectangles contained by the opposite sides are together double of the quadrilateral figure.

66. If a rectangular parallelogram be inscribed in a right-angled triangle, and they have the right angle common; the rectangle contained by the segments of the hypothenuse is equal to the sum of the rectangles contained by the segments of the sides about the right angle.

67. If on the diameter of a semicircle two equal circles be described, and in the curvilinear space included by the three circumferences a circle be inscribed; its diameter will be to that of the equal circles in the proportion of two to three.

68. If through the middle point of any chord of a circle two chords be drawn; the lines joining their extremities will intersect the first chord at equal distances from the middle point.

69. The longest side of a trapezium being given, and made the diameter of the circumscribed circle; also the distance between its extremity and the intersection of the opposite side produced to meet it, and the angle formed by the intersection of the diagonals: to construct the trapezium.

70. The diagonals of a quadrilateral figure inscribed in a circle are to one another as the sums of the rectangles of the sides which meet their extremities.

71. The square described on the side of an equilateral and equiangular pentagon inscribed in a circle, is equal to the sum of the squares of the side of a regular hexagon and decagon inscribed in the same circle.

72. If the opposite sides of an irregular hexagon inscribed in a circle be produced till they meet; the three points of intersection will be in the same straight line.

SECTION VII. Page 232.

1. The vertical angle of an oblique-angled triangle inscribed in a circle, is greater or less than a right angle, by the angle contained by the base and the diameter drawn from the extremity of the base.

2. If from the vertex of an isosceles triangle a circle be described with a radius less than one of the equal sides, but greater than the perpendicular; the parts of the base cut off by it will be equal.

3. If a circle be inscribed in a right-angled triangle; the difference between the two sides containing the right angle and the hypothenuse, is equal to the diameter of the circle.

4. If a semicircle be inscribed in a right-angled triangle so as to touch the hypothenuse and perpendicular, and from the extremity of its diameter a line be drawn through the point of contact, to meet the perpendicular produced; the part produced will be equal to the perpendicular.

5. If the base of any triangle be bisected by the diameter of its circumscribing circle, and from the extremity of that diameter a perpendicular be let fall upon the longer side; it will divide that side into segments, one of which will be equal to half the sum, and the other to half the difference of the sides.

6. The same supposition being made as in the last proposition; if from the point where the perpendicular meets the longer side, another perpendicular be let fall on the line bisecting the vertical angle; it will pass through the middle of the base.

7. If a point be taken without a circle, and from it tangents be drawn to the circle, and another point be taken in the circumference between the two tangents, and a tangent be drawn to it; the sum of the sides of the triangle thus formed is equal to the sum of the two tangents.

8. Of all triangles on the same base and between the same parallels, the isosceles has the greatest vertical angle.

Cor. Of all triangles on the same base, and having the same vertical angle, the isosceles is the greatest.

9. If through the vertex of an equilateral triangle a perpendicular be drawn to the side, meeting a perpendicular to the base drawn

from its extremity; the line intercepted between the vertex and the latter perpendicular is equal to the radius of the circumscribing circle.

10. If a triangle be inscribed in a semicircle, and a perpendicular drawn from any point in the diameter meeting one side, the circumference, and the other side produced; the segments cut off will be in continued proportion.

11. If a triangle be inscribed in a semicircle, and one side be equal to the semidiameter; the other side will be a mean proportional between that side and a line equal to that side and the diameter together.

12. If a circle be inscribed in a right-angled triangle; to determine the least angle that can be formed by two lines drawn from the extremity of the hypothenuse to the circumference of the circle.

13. If an equilateral triangle be inscribed in a circle, and through the angular points another be circumscribed; to determine the ratio which they bear to each other.

14. A straight line drawn from the vertex of an equilateral triangle inscribed in a circle to any point in the opposite circumference, is equal to the two lines together, which are drawn from the extremities of the base to the same point.

15. If the base of a triangle be produced both ways, so that each part produced may be equal to the adjacent side, and through the extremities of the parts produced and the vertex a circle be described; the line joining its centre and the vertex of the triangle will bisect the angle at the vertex.

16. If an isosceles triangle be inscribed in a circle, and from the vertical angle a line be drawn meeting the circumference and the base; either equal side is a mean proportional between the segments of the line thus drawn.

17. If from the extremities of one of the equal sides of an isosceles triangle inscribed in a circle, tangents be drawn to the circle and produced to meet; two lines drawn to any point in the circumference from the point of concourse and one point of contact, will divide the base (produced if necessary) in geometrical proportion.

18. If on the sides of a triangle segments of circles be described

similar to a segment on the base, and from the extremities of the base tangents be drawn intersecting their circumferences; the points of intersection and the vertex of the triangle will be in the same straight line.

19. The centre of the circle which will touch two semicircles described on the sides of a right-angled triangle is in the middle point of the hypothenuse.

Cor. Its diameter will be equal to the sides together.

20. If on the three sides of a right-angled triangle semicircles be described, and with the centres of those described on the sides, circles be described touching that described on the base; they will also touch the other semicircles.

21. If from any point in the circumference of a circle perpendiculars be drawn to the sides of the inscribed triangle; the three points of intersection will be in the same straight line.

22. The base of a right-angled triangle not being greater than the perpendicular; if on any line drawn from the vertex to the base a semicircle be described, and a chord equal to the perpendicular placed in it, and bisected; the point of bisection will always fall within the triangle.

23. The straight line bisecting any angle of a triangle inscribed in a given circle, cuts the circumference in a point, which is equidistant from the extremities of the side opposite to the bisected angle, and from the centre of a circle inscribed in the triangle.

24. The perpendicular from the vertex on the base of an equilateral triangle is equal to the side of an equilateral triangle inscribed in a circle whose diameter is the base.

25. If an equilateral triangle be inscribed in a circle, and the adjacent arcs cut off by two of its sides be bisected; the line joining the points of bisection will be trisected by the sides.

26. If any triangle be inscribed in a circle, and from the vertex a line be drawn parallel to a tangent at either extremity of the base; this line will be a fourth proportional to the base and two sides.

27. If a triangle be inscribed in a circle, and from its vertex lines be drawn parallel to tangents at the extremities of its base; they will cut off similar triangles.

COR. 1. The rectangle contained by the segments of the base adjacent to the angles is equal to the square of either line drawn from the vertex.

COR. 2. Those segments are also in the duplicate ratio of the adjacent sides.

28. If one circle be circumscribed and another inscribed in a given triangle, and a line be drawn from the vertical angle to the centre of the inner, and produced to the circumference of the outer circle; the whole line thus produced has to the part produced the same ratio that the sum of the sides of the triangle has to the base.

29. If in a right-angled triangle, a perpendicular be drawn from the right angle to the hypothenuse, and circles inscribed within the triangles on each side of it; their diameters will be to each other as the subtending sides of the right-angled triangle.

30. To find the locus of the vertex of a triangle, whose base and ratio of the other two sides are given.

31. A given straight line being divided into any three parts; to determine a point such, that lines drawn to the points of section and to the extremities of the line shall contain three equal angles.

32. If two equal lines touch two unequal circles, and from the extremities of them lines containing equal angles be drawn cutting the circles, and the points of section joined; the triangles so formed will be reciprocally proportional.

33. If from an angle of a triangle a line be drawn to cut the opposite side, so that the rectangle contained by the sides including the angle be equal to the rectangle contained by the segments of the side together with the square of the line so drawn; that line bisects the angle.

34. In any triangle, if perpendiculars be drawn from the angles to the opposite sides, they will all meet in a point.

35. If from the extremities of the base of any triangle, two perpendiculars be let fall on the line bisecting the vertical angle; and through the points where they meet that line, and the point in the base, whereon the perpendicular from the vertical angle falls, a circle be described; that circle will bisect the base of the triangle.

36. If from one of the angles of a triangle a straight line be

drawn through the centre of its inscribed circle, and a perpendicular be drawn to this line from one of the other angles; the point of intersection of the perpendicular, and the two points of contact of the inscribed circle which are adjacent to the remaining angle, are in the same straight line.

37. If from the three angles of any triangle three straight lines be drawn to the points where the inscribed circle touches the sides; these lines shall intersect each other in the same point.

38. If three circles touch each other, two of which are equal; the vertical angle of the triangle formed by joining the points of contact, is equal to either of the angles at the base of the triangle, which is formed by joining their centres.

39. If three equal circles touch each other; to compare the area of the triangle formed by joining their centres with the area of the triangle formed by joining the points of contact.

40. If four straight lines intersect each other, and form four triangles; the circles which circumscribe them will pass through one and the same point.

41. Having given the base and vertical angle of a triangle; to determine the locus of the extremity of the line, which always bisects the vertical angle, and is equal to half the sum of the sides containing that angle.

42. If from the extremities of the base of a triangle inscribed in a circle, perpendiculars be drawn to the opposite sides, intersecting a diameter which is perpendicular to the base; the segments of the diameter intercepted between these points and a point in it, whose distance from the base is equal to the lesser segment of the diameter made by the base, will be to one another in the ratio of the sides of the triangle.

43. If the exterior angle of a triangle be bisected by a straight line which cuts the base produced; the square of the bisecting line is equal to the difference of the rectangles of the segments of the base and of the sides of the triangle.

SECTION VIII. Page 263.

1. If from the centre of a circle a line be drawn to any point in the chord of an arc; the square of that line together with the rectangle contained by the segments of the chord will be equal to the square described on the radius.

2. If two straight lines in a circle cut each other at right angles; the sums of the squares of the two lines joining their extremities will be equal.

3. If two points be taken in the diameter of a circle, equidistant from the centre; the sum of the squares of the two lines drawn from these points to any point in the circumference will be always the same.

4. If from any point in the diameter of a semicircle there be drawn two straight lines to the circumference, one to its point of bisection, and the other at right angles to the diameter; the squares of these two lines are together double of the square of the semidiameter.

5. If a straight line be drawn at right angles to the diameter of a circle, and be cut by any other line; the rectangle contained by the segments of this cutting line, together with the square of that part of the perpendicular line which is intercepted between it and the diameter, is always of the same magnitude.

6. A straight line being drawn from the centre of a quadrant, bisecting the arc and meeting a tangent drawn from one extremity: if from any point in the bounding radius a line be drawn parallel to the tangent; the sum of the squares of the segments of it, cut off by the aforesaid line and by the circumference will be equal to the square of the radius.

7. If from a point without a circle there be drawn two straight lines, one of which is perpendicular to a diameter, and the other cuts the circle; the square of the perpendicular is equal to the rectangle contained by the whole cutting line and the part without the circle, together with the rectangle contained by the segments of the diameter.

8. If any straight line be drawn perpendicular to the diameter of a given circle, and produced to cut any chord; the rectangle con-

tained by the segments of the diameter will be less or greater than the rectangle contained by the segments of the chord, by the square of the line intercepted between them, according as it is drawn without or within the circle.

9. If a diameter of a circle be produced to bisect a line at right angles, the length of which is the double of a mean proportional between the whole line through the centre and the part without the circle; and from any point in the double of the mean proportional a line be drawn cutting the circle; the sum of the squares of the segments of the double mean proportional will be equal to twice the rectangle contained by this cutting line and the part without the circle.

10. If from a point without a circle two straight lines be drawn, one through the centre to the circumference, and the other perpendicular to it, and on the former a mean proportional be taken between the whole line and the part without the circle; any other line passing through that extremity of the mean proportional which is within the circle, and terminated by the circumference and perpendicular, will be similarly divided.

11. If a chord be drawn parallel to the diameter of a circle, and from any point in the diameter lines be drawn to its extremities; the sum of their squares will be equal to the sum of the squares of the segments of the diameter.

12. If through a point within or without a circle, two straight lines be drawn at right angles to each other, and meeting the circumference; the squares of the segments of them are together equal to the square of the diameter.

13. If from a point without a circle there be drawn two straight lines, one of which touches the circle and the other cuts it, and from the point of contact a perpendicular be drawn to the diameter; the square of the line which touches the circle is equal to the square of that part of the cutting line which is intercepted by the perpendicular, together with the rectangle contained by the segments of that part of it which is within the circle.

14. A straight line drawn from the concourse of two tangents to the concave circumference of a circle is divided harmonically by the convex circumference and the chord which joins the points of contact.

15. If from the extremities of any chord in a circle straight lines be drawn to any point in the circumference meeting a diameter perpendicular to the chord; the rectangle contained by the distances of their points of intersection from the centre is equal to the square described upon the radius.

16. If from any point in the base, or base produced, of the segment of a circle, a line be drawn making therewith an angle equal to the angle in the segment, and from the extremity of the base any line be drawn to the former, and cutting the circumference; the rectangle contained by this line and the part of it within the segment is always of the same magnitude.

17. To determine the locus of the extremities of any number of straight lines drawn from a given point, so that the rectangle contained by each and a segment cut off from each by a line given in position may be equal to a given rectangle.

18. If from a given point two straight lines be drawn containing a given angle, and such that their rectangle may be equal to a given rectilineal figure, and one of them be terminated by a straight line given in position; to determine the locus of the extremity of the other.

19. If from the vertical angle of a triangle two lines be drawn to the base making equal angles with the adjacent sides; the squares of those sides will be proportional to the rectangles contained by the adjacent segments of the base.

20. If a line placed in one circle be made the diameter of a second, the circumference of the latter passing through the centre of the former, and any chord in the former circle be drawn through this diameter perpendicularly; the rectangle contained by the segments made by the circumference of the latter circle will be equal to that contained by the whole diameter and a mean proportional between its segments.

21. If semicircles be described on the segments of the base made by a perpendicular drawn from the right angle of a triangle; they will cut off from the sides, segments which will be in the triplicate ratio of the sides.

22. If from any point in the diameter of a semicircle a perpendicular be drawn, and from the extremities of the diameter lines be

drawn to any point in the circumference, and meeting the perpendicular; the rectangle contained by the segments which they cut off from the perpendicular, will be equal to the rectangle contained by the segments of the diameter.

23. If from the point of bisection and any other point in a given arc of a circle, two parallel lines be drawn, the former terminated by the circumference, the latter by the chord of the arc; the rectangle contained by these two lines will be equal to that contained by the lines which join the latter point with each extremity of the given arc.

24. If two circles cut each other, and from either point of intersection lines be drawn meeting both circumferences; the rectangles contained by the segments of these lines are to one another in the ratio of the perpendiculars drawn from their intersection with the inner circumferences upon the line joining the intersections of the circles.

25. If on opposite sides of any point in the chord of a circle, two lines be taken, one terminating in the chord the other in the chord produced, whose rectangle is equal to that contained by the segments of the chord; and the extremities of the lines so taken be joined to those of any other chord passing through the same point; the line joining their intersections of the circle will be parallel to the first chord.

26. If from two points without a circle two tangents be drawn, the sum of the squares of which is equal to the square of the line joining those points; and from one of them a line be drawn cutting the circle, and two lines from the other point to the intersections with the circumference; the points in which these two lines cut the circle, are in the same straight line with the former point.

27. If from the vertex of a triangle there be drawn a line to any point in the base, from which point lines are drawn parallel to the sides; the sum of the rectangles of each side and its segment adjacent to the vertex will be equal to the square of the line drawn from the vertex, together with the rectangle contained by the segments of the base.

28. If on the chord of a quadrantal arc a semicircle be described; the area of the lune so formed will be equal to the area of the triangle formed by the chord and terminating radii of the quadrant.

29. If from the extremities of the side of a square circles be described with radii equal, the former to the side and the latter to the diagonal of the square; the area of the lune so formed will be equal to the area of the square.

30. If on the sides of a triangle inscribed in a circle, semicircles be described; the two lunes formed thereby will together be equal to the area of the triangle.

31. If on the two longer sides of a rectangular parallelogram as diameters, two semicircles be described towards the same parts; the figure contained by the two remaining sides of the parallelogram and the two circumferences shall be equal to the parallelogram.

32. If two points be taken at equal distances from the extremities of a quadrant, and perpendiculars be drawn from these points to the radius; the mixtilinear space cut off, shall be equal to the sector which stands on the arc between them.

33. If the arc of a semicircle be trisected, and from the points of section lines be drawn to either extremity of the diameter; the difference of the two segments thus made will be equal to the sector which stands on either of the arcs.

34. If a straight line be placed in a circle, and on the radius passing through one extremity, as a diameter, another circle be described; the segments of the two circles cut off by the above straight line will be similar, and in the ratio of four to one.

35. If on any two segments of the diameter of a semicircle semicircles be described; the area included between the three circumferences will be equal to the area of a circle whose diameter is a mean proportional between the segments.

36. If the diameter of a semicircle be divided into any number of parts, and on them semicircles be described; their circumferences will together be equal to the circumference of the given semicircle.

37. If two equal circles cut each other, and from either point of section a line be drawn meeting the two circumferences; the area cut off by the part of this line between the two circumferences will be equal to the area of the triangle contained by that part and lines drawn to its extremities from the other point of section.

38. If two equal circles touch each other externally, and through

the point of contact another be described with the same radius; the area contained by the convex circumferences cut off from the touching circles, and the part of the third without them, is equal to the area of the quadrilateral figure formed by lines drawn from the points of intersection to the point of contact, and to the point where the third circle is cut by a tangent drawn to the point of contact of the two circles.

39. If a straight line be divided into any two parts, and upon the whole and the two parts semicircles be described; and from the point of section a perpendicular be drawn, on each side of which circles are described touching it and the semicircles; these circles will be equal.

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SECTION IX. Page 291.

1. Given one angle, a side adjacent to it, and the difference of the other two sides; to construct the triangle.

2. Given one angle, a side opposite to it, and the difference of the other two sides; to construct the triangle.

3. Given the base, and one of the angles at the base; to construct the triangle when the side opposite to the given angle is equal to half the sum of the other side and a given line.

4. Given the base of a right-angled triangle, and the sum of the hypothenuse and a straight line, to which the perpendicular has a given ratio; to construct the triangle.

5. Given the perpendicular drawn from the vertical angle to the base, and the difference between each side and the adjacent segment of the base made by the perpendicular; to construct the triangle.

6. Given the vertical angle, and the base; to construct the triangle when the line drawn from the vertex cutting the base in any given ratio, bisects the vertical angle.

7. Given the vertical angle, and one of the sides containing it; to construct the triangle, when the line drawn from the vertex making a given angle with the base, bisects the triangle.

8. Given one angle, a side opposite to it, and the sum of the other two sides; to construct the triangle.

9. Given the vertical angle, the line bisecting the base, and the angle which the bisecting line makes with the base; to construct the triangle.

10. Given the vertical angle, the perpendicular drawn from it to the base, and the ratio of the segments of the base made by it; to construct the triangle.

11. Given the vertical angle, the base, and a line drawn from either of the angles at the base to cut the opposite side in a given ratio; to construct the triangle.

12. Given the perpendicular, the line bisecting the vertical angle, and the line bisecting the base; to construct the triangle.

13. Given the line bisecting the vertical angle, the line bisecting the base, and the difference of the angles at the base; to construct the triangle.

14. Given the vertical angle, and the line drawn to the base bisecting the angle, and the difference between the base and the sum of the sides; to construct the triangle.

15. Given the line bisecting the vertical angle, the perpendicular drawn to it from one of the angles at the base, and the other angle at the base; to construct the triangle.

16. Given the line bisecting the vertical angle, and the perpendiculars drawn to that line from the extremities of the base; to construct the triangle.

17. Given the vertical angle, the difference of the two sides containing it, and the difference of the segments of the base made by a perpendicular from the vertex; to construct the triangle.

18. Given the base, and vertical angle; to construct the triangle, when the square of one side is equal to the square of the base, and three times the square of the other side.

19. Given the base and perpendicular; to construct the triangle, when the rectangle contained by the sides is equal to twice the rectangle contained by the segments of the base made by the line bisecting the vertical angle.

20. In a right-angled triangle, having given the sum of the base and hypothenuse, and the sum of the base and perpendicular; to construct the triangle.

21. Given the perimeter of a right-angled triangle whose sides are in geometrical progression; to construct the triangle.

22. Given the difference of the angles at the base, the ratio of the segments of the base made by the perpendicular, and the sum of the sides; to construct the triangle.

23. Given the difference of the angles at the base, the ratio of the sides, and the length of a third proportional to the difference of the segments of the base made by a perpendicular from the vertex and the shorter side; to construct the triangle.

24. Given the base of a right-angled triangle; to construct it, when parts, equal to given lines, being cut off from the hypothenuse and perpendicular, the remainders have a given ratio.

25. Given one angle of a triangle, and the sums of each of the sides containing it and the third side; to construct the triangle.

26. Given the vertical angle, and the ratio of the sides containing it, as also the diameter of the circumscribing circle; to construct the triangle.

27. Given the vertical angle, and the radii of the inscribed and circumscribing circles; to construct the triangle.

28. Given the vertical angle, the radius of the inscribed circle, and the rectangle contained by the straight lines drawn from the centre of that circle to the angles at the base; to construct the triangle.

29. Given the base, one of the angles at the base, and the point in which the diameter of the circumscribing circle drawn from the vertex meets the base; to construct the triangle.

30. Given the vertical angle, the base, and the difference between two lines drawn from the centre of the inscribed circle to the angles at the base; to construct the triangle.

31. Given that segment of the line bisecting the vertical angle which is intercepted by perpendiculars let fall upon it from the angles at the base; the ratio of the sides; and the ratio of the radius of the

inscribed circle to the segment of the base which is intercepted between the line bisecting the vertical angle and the point of contact of the inscribed circle; to construct the triangle.

32. Given the line bisecting the vertical angle, and the differences between each side and the adjacent segment of the base made by the bisecting line; to construct the triangle.

33. Given one of the angles at the base, the side opposite to it, and the rectangle contained by the base and that segment of it made by the perpendicular which is adjacent to the given angle; to construct the triangle.

34. Given the vertical angle, and the lengths of two lines drawn from the extremities of the base to the points of bisection of the sides; to construct the triangle.

35. Given the lengths of three lines drawn from the angles to the points of bisection of the opposite sides; to construct the triangle.

36. Given the segments of the base made by the perpendicular, and one of the angles at the base triple the other; to construct the triangle.

37. The area and hypothenuse of a right-angled triangle being given; to construct the triangle.

38. Given one angle, and a line drawn from one of the others bisecting the side opposite to it; to construct the triangle, when the area is also given.

39. In two similar right-angled triangles, the sum of the base of one and perpendicular of the other is given; to determine the triangles such that their hypothenuses may contain the right angle of another triangle similar to them, and the sum of the three areas may be equal to a given area.

40. Given the vertical angle, the area, and the distance between the centres of the inscribed circle and the circle which touches the base and the two sides produced; to construct the triangle.

41. Given the area, the line from the vertex dividing the base into segments which have a given ratio, and either of the angles at the base; to construct the triangle.

42. Given the difference between the segments of the base made by the perpendicular, the sum of the squares of the sides, and the area; to construct the triangle.

43. Given the base, one of the angles at the base, and the difference between the side opposite to it and the perpendicular; to construct the triangle.

44. Given the vertical angle, the difference of the base and one side, and the sum of the perpendicular drawn from the angle at the base contiguous to that side upon the opposite side and the segment cut off by it from that opposite side contiguous to the other angle at the base; to construct the triangle.

45. Given the base, the difference of the sides, and the segment intercepted between the vertex and a perpendicular from one of the angles at the base upon the opposite side; to construct the triangle.

46. Given the vertical angle, the side of the inscribed square, and the rectangle contained by one side and its segment adjacent to the base made by the angular point of the inscribed square; to construct the triangle.

◆

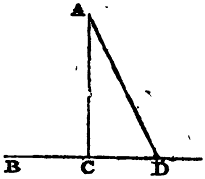
GEOMETRICAL

PROBLEMS.

Angles and straight lines

SECT. I.

(1.) *FROM a given point, to draw the shortest line possible to a given straight line.*

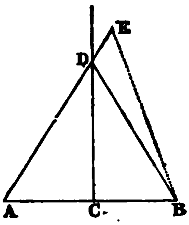


Let *A* be the given point, and *BD* the given line. From *A* let fall the perpendicular *AC*; this will be less than any other line *AD* drawn from *A* to *BD*.

For since *AC* is perpendicular to *BD*, the angle *ACD* is a right angle, therefore the angle *ADC* is less than a right angle (Eucl. i. 32.) and consequently less than *ACD*. But the greater angle is subtended by the greater side (Eucl. i. 19.); therefore *AD* is greater than *AC*. In the same manner every other line drawn from *A* to *BD* may be shewn to be greater than *AC*; therefore *AC* is the least.

* * *

(2.) *If a perpendicular be drawn bisecting a given straight line; any point in this perpendicular is at equal* *distances, and any point without the perpendicular is at unequal distances from the extremities of the line.*



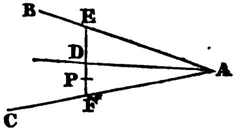
From *C* the point of bisection let *CD* be drawn at right angles to *AB*; any point *D* is at equal distances from *A* and *B*.

Join *AD*, *DB*. Since *AC* = *CB* and *CD* is common, and the angle *ACD* = *BCD* being right angles, *AD* = *DB*. And the same may be proved of lines drawn from any other point in *CD* to *A* and *B*.

But if a point *E* be taken which is not in *CD*, join *EA* cutting the perpendicular in *D*; join *EB*, *DB*. Then *AD* = *DB* from the first part, and *AE* is equal to *AD*, *DE*, that is, to *BD*, *DE*, and is therefore greater than *BE*, (Eucl. i. 20.); therefore, &c.

* * *

(3.) *Through a given point to draw a straight line which shall make equal angles with two straight lines given in position.*

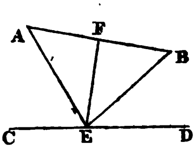


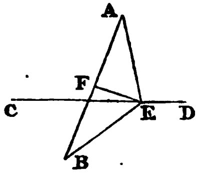
Let *P* be the given point, and *BE*, *CF* the lines given in position. Produce *BE*, *CF* to meet in *A*, and bisect the angle *BAC* by the line *AD*. From *P* let fall the perpendicular *PD*, and produce it both ways to *E* and *F*. It will be the line required.

For the angle *EAD* is equal to the angle *FAD*, the angles at *D* right angles, and *AD* common, therefore (Eucl. i. 26.) the angle *AED* is equal to the angle *AFD*; therefore, &c.

* * *

(4.) *From two given points to draw two equal straight lines which shall meet in the same point of a line given in position.*



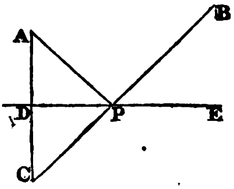


Let A and B be the given points, and CD the given straight line. Join AB, and bisect it in F, and from F draw FE at right angles to AB meeting CD in E, E is the point required.

Join AE, EB. Since $AF = FB$, and FE is common, and the angles at F are right angles, therefore $AE = EB$.

* * *

(5.) *From two given points on the same side of a line given in position, to draw two lines which shall meet in that line, and make equal angles with it.*

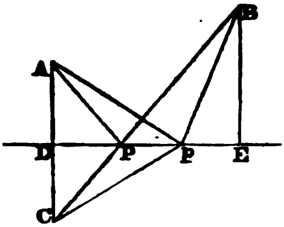


Let A and B be the given points and DE the line given in position. From A let fall the perpendicular AD, and produce it to C making $DC = AD$. Join CB, AP. AP, PB will be the lines required.

Since $AD = DC$ and DP is common, and the angles at D are right angles, therefore the triangles APD, CPD are equal, and the angle $APD = CPD =$ the vertically opposite angle BPE.

* * *

(6.) *From two given points on the same side of a line given in position, to draw two lines which shall meet in a point in this line, so that their sum shall be less* *than the sum of any two lines drawn from the same points and terminated at any other point in the same line.*

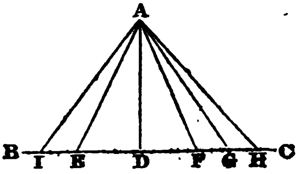


Let A and B be the given points, and DE the line given in position; from A and B let fall the perpendiculars AD, BE, and produce AD to C making $CD = DA$. Join BC cutting DE in P. Join AP; AP and PB shall be less than any other two lines Ap, pB drawn from A and B to any other point p in the line DE.

For $AD = DC$ and DP is common and the angles at D are right angles, $\therefore AP = PC$. In the same manner, if pC be joined, it may be shewn that $Ap = pC$. Hence AP and BP together are equal to BC, and Ap, pB are equal to Cp, pB. Now (Eucl. i. 20.) BC is less than Bp, pC, and therefore AP, PB are less than Ap, pB; therefore, &c.

* * *

(7.) *Of all straight lines which can be drawn from a given point to an indefinite straight line, that which is nearer to the perpendicular is less than the more remote. And from the same point there cannot be drawn more than two straight lines equal to each other, viz. one on each side of the perpendicular.*



Let A be the given point, and BC the given indefinite straight line. From A let fall the perpendicular AD, and draw any other lines AF, AG, AH, &c. of which AF is nearer to AD than AG is, and

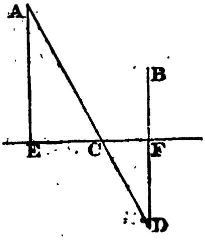
AG than *AH*; then *AF* will be less than *AG* and *AG* than *AH*.

For since the angle at *D* is a right angle, the angle *AFG* is greater than a right angle (Eucl. i. 16.), and therefore greater than *AGF*, hence (Eucl. i. 19.) *AG* is greater than *AF*. In the same manner it may be shewn that *AH* is greater than *AG*.

And from *A* there can only be drawn to *BC* two straight lines equal to each other, viz. one on each side of *AD*. Make *DE*=*DF* and join *AE*. Then *AE*=*AF* (i. 2.). And besides *AE* no other line can be drawn equal to *AF*. For, if possible, let *AI*=*AF*. Then because *AI*=*AF* and *AF*=*AE*, therefore *AI*=*AE*, i. e. a line more remote is equal to one nearer the perpendicular, which is impossible; therefore *AI* is not equal to *AF*. In the same manner it may be shewn that no other but *AE* can be equal to *AF*, therefore, &c.

* * *

(8.) *Through a given point, to draw a straight line, so that the parts of it intercepted between that point and perpendiculars drawn from two other given points may have a given ratio.*



Let *A* and *B* be the points from which the perpendiculars are to be drawn, and *C* the point through which the line is to be drawn. Join *AC*, and produce it to *D*, making *AC* : *CD* in the given ratio; join *BD*, and through *C* draw *ECF* perpendicular to *BD*. *ECF* is the line required.

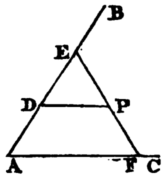
Draw *AE* parallel to *BD*, and ∴ perpendicular to

EF. The triangles ACE, DFC, having each a right angle, and the angles at C equal, are equiangular, whence

$$CE : CF :: AC : CD, \text{ i. e. in the given ratio.}$$

* * *

(9.) *From a given point between two indefinite right lines given in position, to draw a line which shall be terminated by the given lines, and bisected in the given point.*



Let AB, AC be the given lines, meeting in A. From P the given point draw PD parallel to AC one of the lines, and make $DE = DA$. Join EP and produce it to F, then will EF be bisected in P.

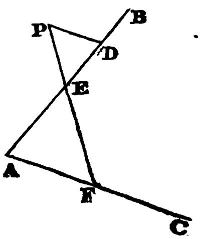
For since DP is parallel to AF, (Eucl. vi. 2.)

$$EP : PF :: ED : DA, \text{ i. e. in a ratio of equality.}$$

Cor. If it be required to draw a line through P which shall be terminated by the given lines, and divided in any given ratio in P, draw PD parallel to AC and take $AD : DE$ in the given ratio and draw EPF, it will be the line required.

* * *

(10.) *From a given point without two indefinite right lines given in position; to draw a line such that the parts intercepted by the point and the lines may have a given ratio.*



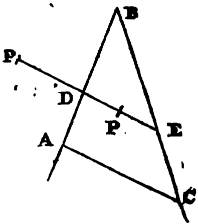
Let AB, AC be the given lines, and P the given point. Draw PD parallel to AC, and take $AD : DE$ in the given ratio. Join PE and produce it to F. Then $PF : PE$ will be in the given ratio.

For the triangles PDE and AEF are similar, having the angles at E equal, as also the angles PDE, EAF, (Eucl. i. 39.)

$$\therefore FE : EP :: AE : ED$$
and *comp.* $PF : PE :: AD : DE$, i. e. in the given ratio.

* * *

(11.) *From a given point to draw a straight line, which shall cut off from lines containing a given angle, segments that shall have a given ratio.*

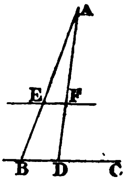


Let ABC be the given angle, and P the given point, either without or within. In BA take any point A, and take $AB : BC$ in the given ratio. Join AC, and from P draw PDE parallel to AC. PDE is the line required.

For since DE is parallel to AC, (Eucl. vi. 2.) $DB : BE :: AB : BC$, i. e. in the given ratio.

* * *

(12.) *If from a given point any number of straight lines be drawn to a straight line given in position, to determine the locus of the points of section which divide them in a given ratio.*

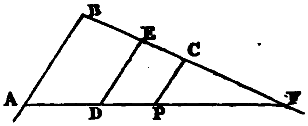


Let A be the given point, and BC the line given in position. From A draw any line AB, and divide it at E in the given ratio; through E draw EF parallel to BD; it is the locus required.

From A draw any other line AD meeting EF in F; then (Eucl. vi. 2.) $AF : FD :: AE : EB$, *i. e.* in the given ratio. In the same manner any other line drawn from A to BD will be divided in the given ratio by EF which therefore is the locus required.

* * *

(13.) *A straight line being drawn parallel to one of the lines containing a given angle and produced to meet the other; through a given point within the angle, to draw a line cutting the other three, so that the part intercepted between the two parallel lines may have a given ratio to the part intercepted between the given point and the other line.*



Let ABC be the given angle, DE parallel to AB, and P the given point.

From P draw PC parallel to DE or AB, and take $BE : CF$ in the given ratio. Join FP and produce it to A; APF is the line required.

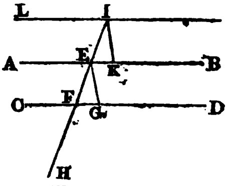
For since DE and CP are parallel to AB,

$$AD : EB :: DF : EF :: PF : CF$$

$\therefore$ *alt.* $AD : PF :: EB : CF$ i. e. in the given ratio.

* * *

(14.) *Two parallel lines being given in position; to draw a third, such that, if from any point in it lines be drawn at given angles to the parallel lines, the intercepted parts may have a given ratio.*



Let AB, CD be the given parallel lines; in AB take any point E; and draw EF, EG making angles equal to the given angles; produce EF, and take $EH : EG$ in the given ratio; produce FE to I so that $FI : IE :: HE : EF$; through I draw LI parallel to AB, it is the line required.

Draw IK parallel to EG; then the triangles IEK, EFG are equiangular,

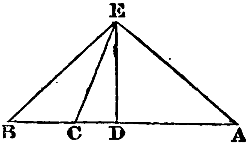
$$\therefore IE : IK :: EF : EG$$

$$\text{but } FI : IE :: HE : EF;$$

$\therefore$ *ex æqu.* $FI : IK :: HE : EG$, *i. e.* in the given ratio; and $IKE = EGF$ which is one of the given angles, and by construction IFG is equal to the other. Also lines drawn from any point in LI, making with AB and CD angles equal to the given angles will be parallel and equal to FI, IK, and $\therefore$ in the given ratio.

* * *

(15.) *If three straight lines drawn from the same point and in the same direction be in continued proportion, and from that point also a line equal to the mean proportional be inclined at any angle; the lines joining the extremity of this line and of the proportionals will contain equal angles.*



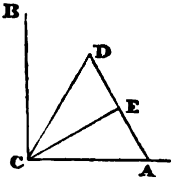
Let $AB : AC :: AC : AD$, and from A let AE be drawn equal to AC, inclined at any angle to AB; join EB, EC, ED; the angle BEC is equal to the angle CED.

For since $AB : AC :: AC : AD$, and $AE = AC$;

$\therefore AB : AE :: AE : AD$, *i. e.* the sides about the angle A are proportional, and $\therefore$ the triangles AEB, AED are similar, and the angle AED is equal to EBA. Also since $CA = AE$, the angle $AEC = ECA$; but ECA is equal to the two BEC, EBC, (Eucl. i. 32.) $\therefore$ also AEC is equal to the two BEC, EBC; of which DEA is equal to EBC; $\therefore$ the remainder DEC is equal to the remainder BEC.

* * *

(16.) *To trisect a right angle.*

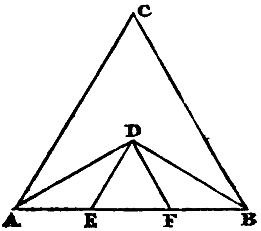


Let *ACB* be a right angle. In *CA* take any point *A*, and on *CA* describe an equilateral triangle *ACD*, and bisect the angle *DCA* by the straight line *CE*; the angles *BCD, DCE, ECA* are equal to one another.

For the angle *DCA* being one of the angles of an equilateral triangle is one third of two right angles, and therefore equal to two thirds of a right angle *BCA*; consequently *BCD* is one third of *BCA*; and since the angle *DCA* is bisected by *CE*, the angles *DCE, ECA* are each of them equal to one third of a right angle, and are therefore equal to *BCD* and to each other.

* * *

(17.) *To trisect a given finite straight line.*

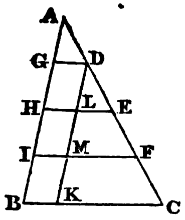


Let *AB* be the given straight line. On it describe an equilateral triangle *ABC*; bisect the angles *CAB, CBA* by the lines *AD, BD* meeting in *D*, and draw *DE, DF* parallel to *CA* and *CB* respectively. *AB* will be trisected in *E* and *F*.

Because *ED* is parallel to *AC*, the angle $EDA = DAC = DAE$ and therefore $AE = ED$. For the same reason $DF = FB$. But *DE* being parallel to *CA* and *DF* to *CB*, the angle *DEF* is equal to the angle *CAB*, and *DFE* to *CBA*, and therefore $EDF = ACB$; and hence the triangle *EDF* is equiangular, and consequently equilateral; therefore $DE = EF = FD$, and hence $AE = EF = FB$, and *AB* is trisected.

* * *

(18.) *To divide a given finite straight line into any number of equal parts.*

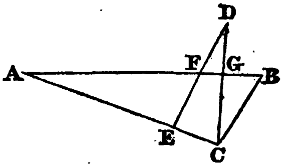


Let AB be the given straight line. Let AC be any other indefinite straight line making any angle with AB, and in it take any point D, and take as many lines DE, EF, FC &c. each equal to AD as the number of parts into which AB is to be divided. Join CB, and draw DG, EH, FI &c. parallel to BC; and therefore parallel to each other; and draw DK parallel to AB.

Then because GD is parallel to HE one of the sides of the triangle AHE, $AG : GH :: AD : DE$; hence $AG = GH$. For the same reason $DL = LM$. But DM being parallel to GI, and DG, LH to MI, the figures DH, HM are parallelograms; therefore $DL = GH$ and $LM = HI$, consequently $GH = HI$. In the same manner it may be shewn that $HI = IB$; and so on, if there be any other parts; therefore AG, GH, HI, IB, &c. are all equal, and AB is divided as was required.

Cor. If it be required to divide the line into parts which shall have a given ratio; take AD, DE, EF, &c in the given ratio, and proceed as in the proposition.

(19.) *To divide a given finite straight line harmonically.*



Let AB be the given straight line. From B draw any straight line BC, and join AC; and from any point E in AC draw ED parallel to CB, and make $FD =$

FE, join DC cutting AB in G. AB is harmonically divided in G and F.

Since BC is parallel to FD, the angle BCG is equal to GDF and the vertically opposite angles at G are equal; therefore the triangles DGF, BGC are similar,

and $BC : BG :: FD : FG$

But FE being parallel to BC,

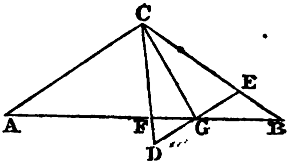
(Eucl. vi. 2.) $AB : BC :: AF : FE = FD$.

$\therefore$ *ex æquali*, $AB : BG :: AF : FG$

or $AB : AF :: BG : FG$.

* * *

(20.) *If a given finite straight line be harmonically divided, and from its extremities and the points of division lines be drawn to meet in any point, so that those from the extremities of the second proportional may be perpendicular to each other, the line drawn from the extremity of this proportional will bisect the angle formed by the lines drawn from the extremities of the other two.*



Let the straight line AB be divided harmonically in the points G and F, and let the lines AC, BC, GC, FC be drawn to any point C so that GC may be perpendicular to CA, the angle BCF will be bisected by CG.

Through G draw EGD parallel to CA meeting CF in D, then EG being parallel to AC, the triangles EGB, ACB are similar; as also the triangles ACF, DFG, hence

$AF : AC :: FG : DG$

but $AB : AF :: GB : GF$,

$\therefore$ *ex æquo* $AB : AC :: BG : GD$.

But *AB* : *AC* :: *GB* : *GE*

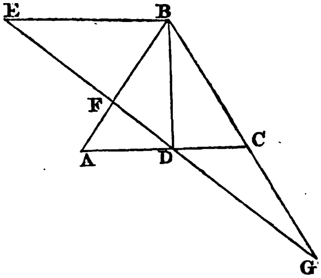
∴ (Eucl. v. 15.) *BG* : *GD* :: *BG* : *GE*,

and therefore *GD* = *GE*, and *GC* is common, and the angles at *G* are right angles, therefore the angle *DCG* = *GCE*, and *FCB* is bisected by *CG*.

* * *

(21.) *If a straight line be drawn through any point in the line bisecting a given angle, and produced to cut the sides containing that angle, as also a line drawn from the angle perpendicular to the bisecting line; it will be harmonically divided.*

Let the angle *ABC* be bisected by the line *BD*, and through any point *D* in this line draw *GDFE* meeting the sides in *G* and *F*, and *BE* a perpendicular to *BD* in *E*; then will *EG* : *EF* :: *GD* : *FD*.



For through *D* draw *AC* parallel to *BE* and therefore perpendicular to *BD*, then the angles *ADB*, *CDB* being right angles are equal, and *ABD* = *CBD* and *BD* is common to the triangles *ADB*, *CDB*, ∴ *AD* = *DC*. But *DC* being parallel to *EB*,

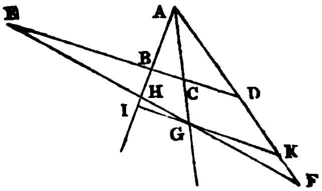
EG : *GD* :: *EB* : *DC* :: *EB* : *AD* :: *EF* : *FD*,

since the triangles *EFB* and *AFD* are similar,

∴ *EG* : *EF* :: *GD* : *FD*.

* * *

(22.) *If from a given point there be drawn three straight lines forming angles less than right angles, and* *from another given point without them a line be drawn intersecting the others so as to be harmonically divided; then will all lines drawn from that point meeting the three lines be harmonically divided.*



From A let AB, AC, AD be drawn making each of the angles BAC, CAD less than a right angle, and from a given point E let EBD be drawn so as to be harmonically divided in C and B; then will any other line EF be harmonically divided in G and H.

Through G draw IK parallel to BD,

then $DC : CB :: KG : GI$,

But $DC : CB :: DE : EB$

$\therefore$ (Eucl. v. 15.) $DE : EB :: KG : GI$

and *alt.* $DE : KG :: EB : GI$

and since DE is parallel to GK, (Eucl. vi. 2.)

$DE : KG :: EF : FG$

and EB being parallel to GI,

$\therefore EB : GI :: EH : HG$,

whence (Eucl. v. 15.)

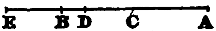
$EF : FG :: EH : HG$

and *alt.* $EF : EH :: FG : HG$.

* * *

(23.) *If a straight line be divided into two equal, and also into two unequal parts, and be produced, so that the part produced may have to the whole line so produced the same ratio that the unequal segments of the line have to each other, then shall the distances of the point of unequal section from one extremity of the given line,*

from its middle point, from the extremity of the part produced, and from the other extremity of the given line, be proportionals.



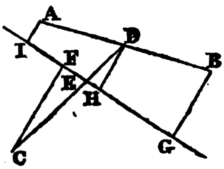
Let AB be divided into two equal parts in C and into two unequal parts in D, and produced to E, so that $BE : EA :: BD : DA$; then will $AD : DC :: ED : DB$.

$$\begin{array}{rl} \text{For since } & BE : EA :: BD : DA \\ \textit{inv.} & AE : EB :: AD : DB \\ \textit{div.} & AB : BE :: 2CD : DB \\ \text{and} & AC : BE :: CD : DB \\ \textit{alt.} & AC : CD :: BE : BD \\ \therefore \textit{comp.} & AD : DC :: ED : DB. \end{array}$$

Cor. The converse may easily be proved to be true.

* * *

(24.) *Three points being given; to determine another, through which if any straight line be drawn, perpendiculars upon it from two of the former, shall together be equal to the perpendicular from the third.*



Let A, B, C be the three given points. Join AB and bisect it in D. Join CD, from which cut off DE equal to a third part of it. E is the point required.

Through E, let any line FG be drawn, and let fall on it the perpendiculars AI, BG, DH, CF; then the angles at F and H being right angles, and the vertical angles at E equal, the triangles CFE, DHE are equiangular,

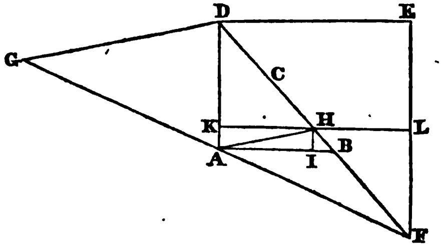
$\therefore FC : DH :: CE : ED :: 2 : 1,$

$\therefore FC = 2DH$; but since AI, BG, DH are parallel, and $AD = DB$, $\therefore AI + BG = 2DH = FC$.

* * *

(25.) *From a given point in one of two straight lines given in position, to draw a line to cut the other, so that if from the point of intersection a perpendicular be let fall upon the former, the segment intercepted between it and the given point, together with the first drawn line, may be equal to a given line.*

Let AB, BC be the lines given in position, and A the given point. Draw AD perpendicular to AB, and meeting BC in D; draw DE parallel to AB and equal to the given line. And draw EF parallel to AD, meeting CB in F. Join FA and produce it, and from D draw $DG = DE$, meeting FG in G, and draw AH parallel to DG, and let fall the perpendicular HI; AH and AI together are equal to the given line.



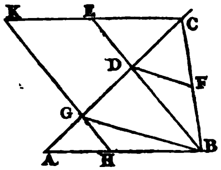
Through H draw KL parallel to DE; then since GD is parallel to AH and HL to DE,

$$\therefore DG : AH :: FD : FH :: DE : HL,$$

$$\text{but } DG = DE, \therefore AH = HL,$$

$$\therefore AH + AI = KL = DE = \text{the given line.}$$

(26.) *One of the lines which contain a given angle, is also given. To determine a point in it such, that if from thence to the finite line there be drawn a line having a given ratio to that segment of it which is adjacent to the given angle; the line so drawn, and the other segment of the given line, may together be equal to another given line.*



Let AB be the given line, and BAC the given angle. From B draw BD to AC, such that it may be to AB in the given ratio*; produce it till $BE =$ the other given line. Through E draw EC parallel to AB, meeting AC in C. Join BC, and draw DF so that it may $= DE$, and draw BG, GH respectively parallel to FD, EB; H is the point required.

For produce HG to meet CE in K;

then (Eucl. vi. 2.) $ED : KG :: CD : CG :: DF : BG$,

but $ED = DF$, $\therefore KG = BG$,

and $HG + GB = HG + GK = BE =$ the given line,

and $HG : HA :: BD : AB$ i. e. in the given ratio.

* * *

(27.) *Two straight lines and a point in each being given in position; to determine the position of another point in each, so that the straight line joining these latter points may be equal to a given line, and their respective distances from the former points in a given ratio.*

Let A and B be the given points in the lines AC, BD which are given in position, and produced to meet

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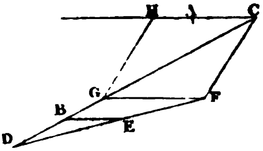
* That is, the given ratio must be less than that of AB to the perpendicular on AD.



in C. Take $BD : AC$ in the given ratio, and from B draw BE parallel and equal to AC. Join DE and produce it to meet CF drawn at any angle from C, equal to the given line; draw FG parallel to EB, and from G draw GH parallel to FC; G and H are the points required.

For BE being parallel to GF,

$DG : GF :: DB : BE$,

or $DG : HC :: DB : AC$,

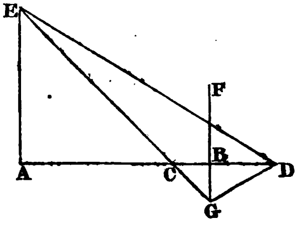
∴ (Eucl. v. 19. Cor.)

$BG : AH :: DB : AC$ in the given ratio,

and $HG = CF =$ the given line.

* * *

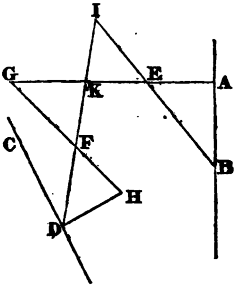
(28.) *If a straight line be divided into any two parts, and produced so that the segments may have the same ratio that the whole line produced has to the part produced, and from the extremities of the given line perpendiculars be erected; then any line drawn through the point of section, meeting these perpendiculars, will be divided at that point into parts, which have the same ratio that those lines have, which are drawn from the extremity of the produced line to the points of intersection with the perpendiculars.*



Let AB be divided into any two parts in C and produced to D so that $AC : CB :: AD : DB$, and from A and B let AE, BF be drawn perpendiculars to AB, and through C let any line ECG be drawn meeting them in E and G, and join DE, DG; then $DE : DG :: CE : CG$.

For because $AC : CB :: AD : DB$
and $EA : BG :: AC : CB$,
(by sim. tri. ACB, BCG)
$\therefore$ (Eucl. v. 11.) $EA : BG :: DA : DB$,
$\therefore$ (Eucl. vi. 6.) the triangles EAD, GDB are equiangular,
and $ED : DG :: AE : BG :: CE : CG$.

(29.) *From two given points, to draw two straight lines which shall contain a given angle, and meet two lines given in position, so that the parts intercepted between those points and the lines may have a given ratio.*

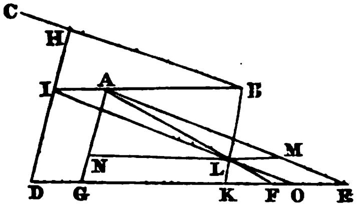


Let AB, CD be the lines given in position, and E, F the given points. From E draw EA perpendicular to AB, and make the angle AGF equal to the given angle. In GF produced take FH such, that the ratio of $EA : FH$ may be the same as the given ratio. Draw HD perpendicular to GH meeting CD in D. Draw DFI and BEI to include the given angle. These are the lines required.

For, since the angles FGE, FIE are equal, as also FKG, EKI, $\therefore$ GFK, IEK or their vertically opposite angles DFH, AEB are equal, and the angles at H and A are right angles, $\therefore$ the triangles FDH, AEB are equiangular, and

$EB : FD :: EA : FH$, i. e. in the given ratio.

(30.) *The length of one of two lines which contain* *a given angle being given; to draw from a given point without them a straight line which shall cut the given line produced, so that the part produced may be in a given ratio to the part cut off from the indefinite line.*



Let AB be the given line, and ABC the given angle; and D the given point. Draw AE, DE parallel to BC, BA respectively; and take $EF : EA$ in the given ratio. Divide DF so that $FE : DG :: FG : AB$. Join AG; and draw DH parallel to AG, and it will be the line cutting BC in H and BA produced in I, as was required.

Join AF; and draw BK parallel to AG cutting AF in L; and draw LM parallel to KE cutting AE in M and AG in N.

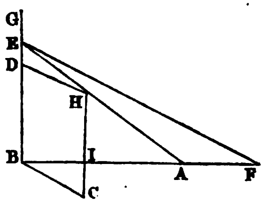
$$\text{Then } FE : LM :: GF : (NL =)\ AB$$
$$\text{and } FE : DG :: FG : AB \text{ by construction;}$$

$\therefore LM = DG = IA$, if therefore ILO be drawn, IL must be equal and parallel to AM, and IO to AE (Eucl. i. 33.) In the same manner it is evident that $HB = IL = AM$; and by similar triangles AFE, ALM,

$$FE : EA :: LM : MA$$
$$:: IA : HB,$$
$$\therefore IA : HB \text{ in the given ratio.}$$

* * *

(31.) *From two given straight lines to cut off two parts, which may have a given ratio; so that the ratio of the remaining parts may also be equal to the ratio of two other given lines.*



Let AB be one of the given lines; draw BG to make any angle with AB, and let BD be equal to the other given line. Take $AB : BE$ in the given ratio of the remaining parts, and $BF : BE$ in the given ratio of the parts to be cut off. Join AE, FE, and draw DH and BC parallel to EF, and HC parallel to DB meeting BC in C, and AB in I.

Then (Eucl. vi. 2.) $AI : IH :: AB : BE$ in the given ratio of the remainders; and the triangles BCI, BFE having the angle $CBI =$ the alternate angle BFE, and $CIB = FBE$, are equiangular,

$$\therefore BI : IC :: BF : BE,$$

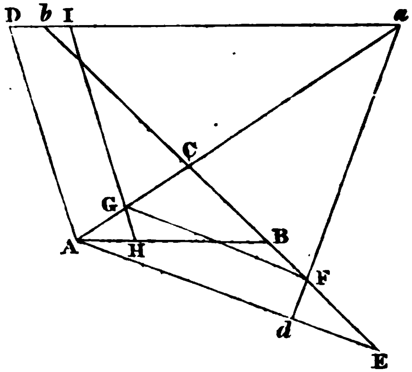
in the ratio of the parts to be cut off; and
AB, HC $(= DB)$ are the given lines.

* * *

(32.) *Three lines being given in position, to determine a point in one of them; from which if two lines be drawn at given angles to the other two, the two lines so drawn may together be equal to a given line.*

Let AB, AC, BC be the three lines given in position, take $AD =$ the given line and making with AB an angle equal to one of the given angles. Through D draw Dba parallel to AB, and meeting AC and BC in a and b. Draw AE to meet CB in E making the angle $AEC =$ the given angle to be made by the line to be drawn, with BC. In AE take $Ad = AD$, and join ad cutting BC in F. Draw FG parallel to EA meeting AC in G, which is the point required.

For through G draw IGH parallel to DA, then the triangles aGI, aAD are similar, and

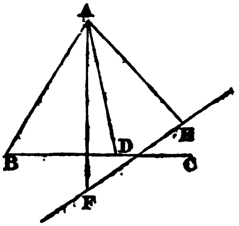


$$aA : AD = Ad :: aG : GI;$$

but $aA : Ad :: aG : GF$,

and $\therefore GI = GF$, $\therefore GH + GF = GH + GI = AD =$ the given line, and the angle $GHB = DAB$ and $GFC = AEC$, $\therefore GHB$, GFC are equal to the given angles.

(33.) ***If from a given point two straight lines be drawn including a given angle, and having a given ratio, and one of them be always terminated by a straight line, given in position, to determine the locus of the extremity of the other.***

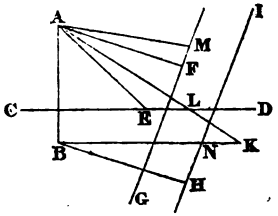


Let A be the given point, and BC the line given in position. From A draw any line AD, and make the angle DAE equal to the given angle, and take AE such that $AD : AE$ may be in the given ratio, and through E draw EF making the angle $AEF = ADB$; EF is the locus required.

Draw any other line *AB*, and make the angle *BAF* $=$*DAE*. Then the angle *BAD*$=$*FAE* and *ADB*$=$ *AEF*, $\therefore$ the triangles *ABD*, *AEF* are equiangular, whence *AB* : *AF* :: *AD* : *AE*, in the given ratio. The same may be proved of any other lines drawn from *A* and containing an angle equal to the given angle, and one of them terminated in *BC*.

* * *

(34.) *If from two given points, straight lines be drawn, containing a given angle and from each of them segments be cut off, having a given ratio; and the extremities of the segments of the lines drawn from one of the points be in a straight line given in position; to determine the locus of the extremities of the segments of lines drawn from the other.*



Let *A* and *B* be the given points, and *CD* the line given in position. From *A* to *CD* draw any line *AE*. Make the angle *EAF*$=$the given angle, and *AE* : *AF* in the given ratio, and let *FG* be the locus of the points *F* (i. 33.). Draw *BH* equal and parallel to *AF*, and through *H* draw *HI* parallel to *GF*. It is the locus required.

Draw any lines *AK*, *BK* containing the angle at *K* $=$the given angle. Make the angle *LAM*$=$the given angle; *AL* : *AM* in the given ratio, and *M* is in the line *GF*. And since *AF* is parallel to *BH*, and *FM* to *HN*, and *BK* to *AM* (since the angles *BKA*, *LAM* are equal) and *AF*$=$*BH*, $\therefore$ the triangles *BHN*, *AFM* are similar and equal, $\therefore$ *AM*$=$*BN*;

but $AL : AM$ is equal to the given ratio,

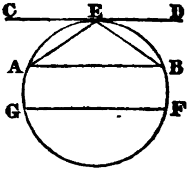
$\therefore$ also $AL : BN$ is equal to the given ratio.

And the same may be proved of any other lines drawn in the same manner.

Straight Lines and Circles

SECT. II.

(1.) *If a straight line be drawn to touch a circle, and be parallel to a chord; the point of contact will be the middle point of the arc cut off by that chord.*



Let CD be drawn touching the circle ABE in the point E, and parallel to the chord AB; E is the middle point of the arc AEB.

Join AE, EB. The angle BAE is equal to the alternate angle CEA, and therefore to the angle EBA in the alternate segment, whence $AE = EB$, and (Eucl. iii. 28.) the arc AE is equal to the arc EB.

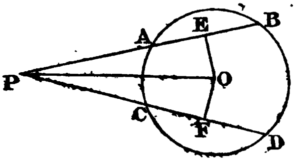
COR. 1. Parallel lines placed in a circle cut off equal parts of the circumference.

If FG be parallel to AB; the arc $EF = EG$, whence $AG = BF$.

COR. 2. The two straight lines in a circle, which join the extremities of two parallel chords are equal to each other. For if AB, FG be parallel, the arcs AG, BF are equal, therefore (Eucl. iii. 29.) the straight lines AG, BF are also equal.

* * *

(2.) *If from a point without a circle, two straight lines be drawn to the concave part of the circumference, making equal angles with the line joining the same point and the centre, the parts of the lines which are intercepted within the circle are equal.*

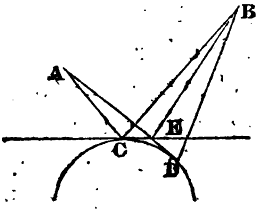


From the point *P* without the circle *ABC* let two lines *PB*, *PD* be drawn making equal angles with *PO*, the line joining *P* and the centre; *AB* shall be equal to *CD*.

Let fall the perpendiculars *OE*, *OF*; then since the angle at *E* is equal to the angle at *F*, and $EPO = FPO$, and the side *PO*, opposite to one of the equal angles in each is common, $\therefore OE = OF$, and consequently (Eucl. iii. 14.) $AB = CD$.

* * *

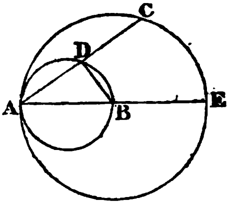
(3.) *Of all straight lines which can be drawn from two given points to meet on the convex circumference of a given circle; the sum of those two will be the least, which make equal angles with the tangent at the point of concourse.*



Let *A* and *B* be two given points, *CE* a tangent to the circle at *C*, where the lines *AC*, *BC* make equal angles with it; and let lines *AD*, *BD* be drawn from *A* and *B* to any other point *D* on the convex circumference; *AC* and *CB* together are less than *AD*, *DB* together.

Let *AD* meet the tangent in *E*. Join *EB*, then (i. 6.) *AC* and *CB* together are less than *AE* and *EB*; but *AE*, *EB* are less than *AD*, *DB* (Eucl. i. 19.), ∴ *a fortiori* *AC*, *CB* are less than *AD*, *DB*. And the same may be proved of lines drawn to every other point in the convex circumference.

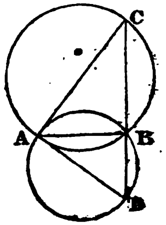
(4.) *If a circle be described on the radius of another circle; any straight line drawn from the point where they meet, to the outer circumference, is bisected by the interior one.*



Let *ADB* be a circle described on the radius *AB* of the circle *ACE*. Draw any line *AC* meeting the circle *ABD* in *D*; *AD* is equal to *DC*.

Join *DB*. Then the angle *ADB* being in a semicircle is a right angle; and therefore *BD* being drawn from the centre *B* of the circle *ACE* bisects *AC* (Eucl. iii. 3.).

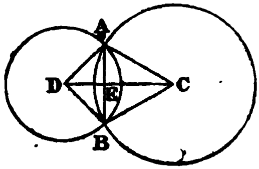
(5.) *If two circles cut each other, and from either point of intersection diameters be drawn; the extremities of these diameters and the other point of intersection shall be in the same straight line.*



Let the two circles *ABC*, *ABD* cut each other in *A* and *B*, draw the diameters *AC*, *AD*, and join *BC*, *BD*; *CB* and *BD* are in the same straight line.

Join *AB*; the angles *ABC*, *ABD* being angles in semicircles are right angles, and therefore (Eucl. i. 13.) *CB* and *BD* are in the same straight line.

(6.) *If two circles cut each other, the straight line joining their two points of intersection, is bisected at right angles by the straight line joining their centres.*



Let the two circles whose centres are C and D cut each other in A and B; join AB, DC. DC bisects AB at right angles.

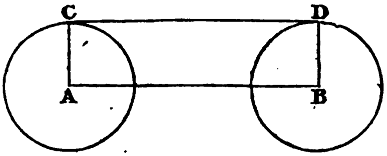
Join BD, DA, AC, CB. Since $AD = DB$ and DC is common to the triangles ADC, BDC and the base $AC = CB$, ∴ (Eucl. i. 8.) the angle $ADE = BDE$. Hence the two sides AD, DE are equal to the two BD, DE and the included angles are equal, ∴ (Eucl. i. 4.) $AE = EB$, and the angle $DEA = DEB$, and being adjacent, they are right angles, i. e. DC bisects AB at right angles.

* * *

(7.) *To draw a straight line which shall touch two given circles.*

1. If the circles be equal.

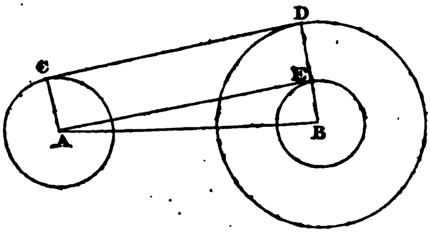
Let A and B be the centres, join AB; and from A



and B draw AC, BD at right angles to it; join CD. Then AC being parallel and equal to DB; CD is parallel to AB, ∴ $CABD$ is a rectangular parallelogram; and the angles at C and D being right angles, CD is a tangent to both circles (Eucl. iii. 16. Cor.).

2. If the circles be unequal, and the line be required to touch them on the same side of the line joining the centres.

Let A and B be the centres; join AB; and with the

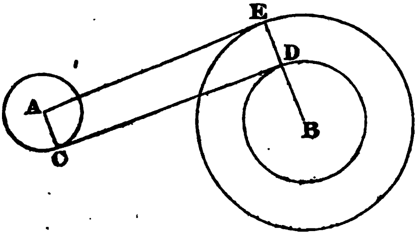


centre B and distance equal to the difference of the given radii, describe a circle, and from A draw AE touching it. Join BE and produce it to D, draw AC parallel to BD, and join CD.

Then AC being parallel and equal to DE, CD is equal and parallel to AE, $\therefore$ $ACDE$ is a parallelogram; and the angle AEB being a right angle, AED is also a right angle; hence the angles at C and D are right angles, and therefore CD touches both circles.

3. If the line be required to touch them on opposite sides of the line joining the centres.

With the centre B and radius equal to the sum of

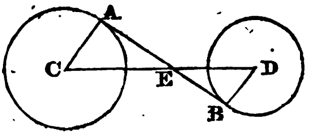


the given radii describe a circle, to which from A draw a tangent AE. Join BE, and let it cut the given circle in D. Draw AC parallel to BE; join CD.

Then *AC* being equal and parallel to *ED*, *ACDE* is a parallelogram, and the angle *AED* being a right angle, the angles at *C* and *D* are right angles, and therefore *CD* touches both circles.

* * *

(8.) *If a line touching two circles cut another line joining their centres, the segments of the latter will be to each other as the diameters of the circles.*



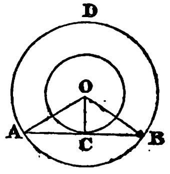
Let the line *AB* touch the circles, whose centres are *C* and *D*, in *A* and *B*, and cut *CD* in the point *E*; *CE* will be to *ED* in the ratio of the diameters of the circles.

Join *CA, BD*. Then the angles at *A* and *B* are right angles and the angles at *E* are vertically opposite, therefore the triangles *AEC*, *BED* are equiangular, and consequently

$$\begin{aligned} CE : ED &:: CA : BD \\ &:: 2CA : 2BD. \end{aligned}$$

* * *

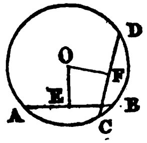
(9.) *If a straight line touch the interior of two concentric circles, and be placed in the outer; it will be bisected at the point of contact.*



Let *AB* touch the interior of two circles, whose common centre is *O*, in the point *C*; *AB* is bisected in *C*.

Join *OC*; then (Eucl. iii. 18.) the angles at *C* are right angles; and *OC* drawn from the centre of the circle *ADB* at right angles to *AB*, bisects it (Eucl. iii. 3.).

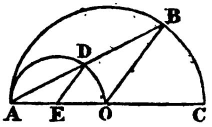
(10.) ***If any number of equal straight lines be placed in a circle; to determine the locus of their points of bisection.***



Let there be any number of lines *AB*, *CD*, placed in the circle whose centre is *O*, and let them be bisected in *E*, *F*; join *OE*, *OF*; then (Eucl. iii. 14.) these lines are equal, and therefore the locus will be a circle whose centre is *O*, and radius equal to the distance of the points of bisection from *O*.

* * *

(11.) ***If from a point in the circumference of a circle, any number of chords be drawn; the locus of their points of bisection will be a circle.***

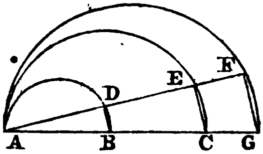


From the given point *A* let any chord *AB* be drawn in the circle, whose centre is *O*; bisect it in *D*. Join *AO*, *BO* and draw *DE* parallel to *BO*.

Then *DE* being parallel to *BO*, the triangles *ADE*, *ABO* are similar, and *BO* is equal *AO*, $\therefore DE = EA$; but $AE : AO :: AD : AB$ (Eucl. vi. 2.), whence $AE = \frac{1}{2} AO$, $\therefore ED = EA = \frac{1}{2} AO$, and the locus will be a circle described on *AO* as a diameter.

* * *

(12.) ***If on the radius of a given semicircle, another semicircle be described, and from the extremity of the diameters any lines be drawn cutting the circumferences, and produced so that the part produced may always have*** *a given ratio to the part intercepted between the two circumferences; to determine the locus of the extremities of these lines.*



On AB the radius of the semicircle AEC let a semicircle ADB be described; and from A draw any line ADE, which produce till $EF : ED$ in the given ratio.

Produce AC to G, making $CG : CB$, in the given ratio, and join DB, EC, FG;

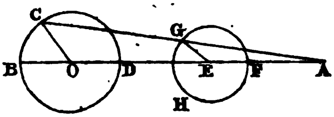
$$\text{then since } FE : ED :: GC : CB,$$

$$\therefore FE : GC :: ED : CB :: DA : AB :: EA : CA,$$

whence (Eucl. vi. 2.) FG is parallel to CE and DB, and the angle AFG is a right angle, and is in a semicircle whose diameter is AG; hence the locus required is a semicircle.

* * *

(13.) *If from a given point without a given circle, straight lines be drawn, and terminated by the circumference; to determine the locus of the points which divide them in a given ratio.*



Let A be the given point and BCD the given circle. Find O its centre and join AO, and divide it in E, so that $AO : AE$ in the given ratio, and find a point F, so that EF may be to OD in the given ratio, and with the centre E and radius EF describe a circle; it will be the locus required.

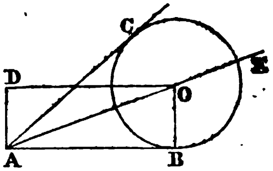
Draw any line AGC; join OC, EG. Since $AO : AE$ in a given ratio, as also $OD : EF$;

$$\therefore OC : EG :: AO : AE,$$

hence *OC* is parallel to *EG*,
and *AC* : *AG* :: *OC* : *EG*, i. e. in the given ratio.

In the same manner it may be shewn that every line drawn from *A* to *BCD* will be divided by the circumference of the circle *GFH* in the same ratio, i.e. *GFH* will be the locus required.

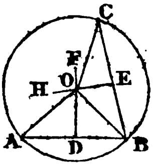
(14.) *Having given the radius of a circle; to determine its centre, when the circle touches two given lines, which are not parallel.*



Let *BA*, *AC* be the two lines which touch the circle, whose radius is given.

Bisect the angle *BAC* by the line *AE*, the centre of the circle will be in this line (Eucl. iv. 4.) From *A* draw *AD* at right angles to *AB* and make it equal to the given radius; through *D* draw *DO* parallel to *AB* meeting *AE* in *O*; then the centre of the circle being in this line also, must be at the point of intersection *O*.

(15.) *Through three given points which are not in the same straight line, a circle may be described; but no other circle can pass through the same points.*

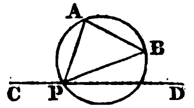


Let *A*, *B*, *C* be the three given points. Join *AB*, *BC*, and bisect them in *D* and *E*; from which points draw *DO*, *EO* at right angles to them; these lines will meet in some point *O*; for if not, they are parallel, and therefore AB, BC must be parallel, which is contrary to the supposition. Join AO, BO, CO.

Since $AD=DB$, and DO is common, and the angles at D equal, $\therefore$ $AO=BO$. In the same manner it may be shewn that $BO=CO$; and the three lines OA, OB, OC being equal, a circle described from the centre O at the distance of any one of them will pass through the extremities of the other two.

And besides this, no other circle can pass through A, B, C: for if it could, its centre would be in DF and EH, and $\therefore$ in their intersection; but two right lines cut each other only in one point, $\therefore$ only one circle can be described.

(16.) *From two given points on the same side of a line given in position, to draw two straight lines which shall contain a given angle, and be terminated in that line.*



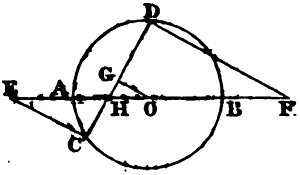
Let A and B be the given points and CD the given line.

Join AB and on it describe a segment of a circle containing an angle equal to the given angle, and (if the problem be possible) meeting CD in P; P is the point required.

For join PA, PB; the angle APB being in the segment is equal to the given angle.

(17.) *If from the extremities of any chord in a circle perpendiculars be drawn, meeting a diameter; the points of intersection are equally distant from the centre.*

At C and D the extremities of the chord CD, let perpendiculars to it be drawn meeting a diameter AB in E and F; E and F are equally distant from the centre O.



Draw OG perpendicular to CD, and therefore bisecting it; then OG is parallel to DF;

$$\text{whence } GD : OF :: HG : HO :: HC : HE$$

since the triangles HGO, HEC are equiangular;

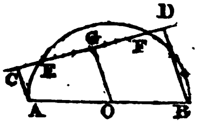
$$\therefore \text{(Eucl. v. 18, 15.) } DG : OF :: GC : OE$$

$$\text{but } GD = GC, \therefore OF = OE.$$

* * *

(18.) *If from the extremities of the diameter of a semicircle perpendiculars be let fall on any line cutting the semicircle; the parts intercepted between those perpendiculars and the circumference are equal.*

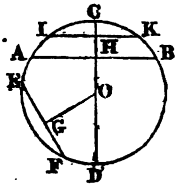
From A and B the extremities of the diameter AB let AC, BD be drawn perpendicular to any line CD cutting the semicircle in E and F; CE is equal to FD.



From O the centre draw OG perpendicular to CD, it will be parallel to AC and BD, whence $CG : GD :: AO : OB$, *i.e.* in a ratio of equality. But (Eucl. iii. 3.) $EG = GF$, and $\therefore CE = FD$.

* * *

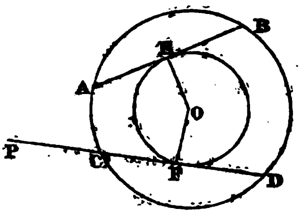
(19.) *In a given circle to place a straight line parallel to a given straight line, and having a given ratio to it.*



Let AB be the given line in the circle ABC whose centre is O. Draw the diameter CD at right angles to AB: and taking a line EF which has to AB the given ratio (Eucl. vi. 12.), place it in the circle ABC; bisect it in G and join OG; make $OH = OG$, and through H, draw IK parallel to AB; IK is the line required.

For since $OG = OH$, $\therefore$ (Eucl. iii. 14.) $IK = EF$, and $EF : AB$ in the given ratio; $\therefore IK : AB$ in the given ratio.

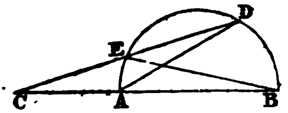
(20.) *Through a given point either without or within a given circle, to draw a straight line, the part of which intercepted by the circle, shall be equal to a given line, not greater than the diameter of the circle.*



Let P be the given point without the circle ABC, whose centre is O. In the circle place a straight line AB equal to the given straight line; which bisect in E; and join OE. With the centre O and radius OE describe a circle; this will touch AB in E, since the angles at E are right angles (Eucl. iii. 3.); from P draw PCD touching the circle in F. PCD is the line required.

Join OF. Then OF being equal to OE, CD will be equal to AB (Eucl. iii. 14.), *i.e.* to the given line.

(21.) *From a given point in the diameter of a semicircle produced, to draw a line cutting the semicircle, so* *that lines drawn from the points of intersection to the extremities of the diameter, cutting each other, may have a given ratio.*

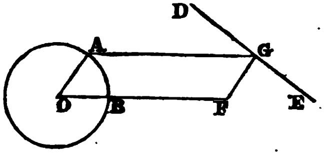


Let C be the given point in the diameter BA produced. Make $BC : CD$ in the given ratio; and from the points E and D, in which CD cuts the semicircle, draw EB, AD to the extremities of the diameter. CD is the line required.

Since the angles EDA, EBA in the same segment are equal, and the angle at C common to the two triangles ACD, CEB, the triangles are equiangular, whence

$BE : AD :: BC : CD$, *i.e.* in the given ratio.

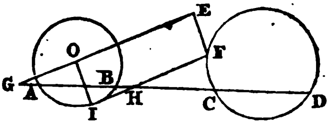
(22.) *From the circumference of a given circle to draw to a straight line given in position, a line which shall be equal and parallel to a given straight line.*



Let AB be the given circle whose centre is O, and DE the line given in position. From O draw OF parallel and equal to the given line; and with the centre F and radius equal to OB, the radius of the given circle describe a circle cutting DE in G: join FG, and draw OA parallel to it; join AG; AG is the line required.

Since $FG = OB = OA$, and is parallel to it, AG is equal and parallel to OF, and $\therefore$ equal and parallel to the given line.

(23.) *The bases of two given circular segments being in the same straight line; to determine a point in it* *such, that a line being drawn through it making a given angle, the part intercepted between the circumferences of the circles may be equal to a given line.*



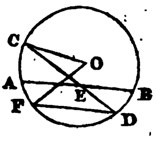
Let *AB, CD*, the bases of the segments be in the same line. Through *O* the centre of the circle *ABI* draw *EOG* making with *AB* an angle equal to the given angle, and make *OE* equal to the given line. From *E* draw *EF*, to the circle *CFD*, equal to the radius *OB*; draw *OI* parallel to *EF*; join *IF* cutting *AD* in *H*; *H* is the point required.

For *OI* being equal and parallel to *EF*, *OE* is equal and parallel to *IF*, ∴ *IF* is equal to the given line; and *IF* being parallel to *EG*, the angle *FHC* is equal to *EGB*, *i. e.* to the given angle.

If the distance of *E* from the centre of the circle *CFD* be less than the sum of the radii there are two points in the circumference *CFD*, and two corresponding parts in *AD* which will answer the conditions.

* * *

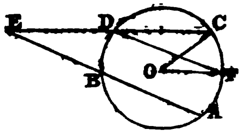
(24.) *If two chords of a given circle intersect each other, the angle of their inclination is equal to half the angle at the centre which stands on an arc equal to the sum or difference of the arcs intercepted between them, according as they meet within or without the circle.*



Let *AB, CD* cut one other in the point *E*; and first within the circle *ABC*; the angle of inclination is equal to half the angle at the centre standing on an arc equal to the sum of *CA* and *DB*.

Through *D* draw *DF* parallel to *BA*. Find *O* the centre of the circle, join *CO*, *FO*. Then *AB* being parallel to *FD*, (ii. 1.) *AF* is equal to *BD*; and the angle *CEA* is equal to *CDF*, *i. e.* to half the angle *COF*, which stands on the arc *CF* equal to *CA* and *BD* together.

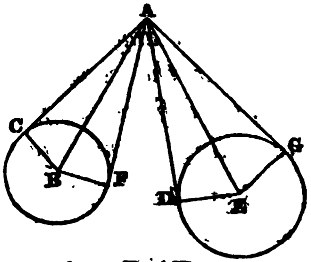
2. Next, let *AB*, *CD* intersect in *E*, without the circle.



The same construction being made, the angle *CEA* is equal to the angle *CDF*, *i. e.* to half *COF*, *i. e.* to half the angle standing on *CF* which is the difference between *CA* and *AF* or *CA* and *BD*.

* * *

(25.) *If from a point without two circles which do not meet each other, two lines be drawn to their centres, which have the same ratio that their radii have; the angle contained by tangents drawn from that point towards the same parts will be equal to the angle contained by lines drawn to the centre.*



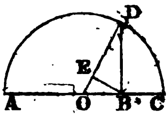
From the point *A* let the lines *AB*, *AE* be drawn to the centres of two circles, and let them have the same ratio that the radii *BC*, *DE*, have; from *A* draw the tangents *AC*, *AD*; as also *AF*, *AG*; each of the angles *CAD*, *FAG* will be equal to *BAE*.

Since *AB* : *BC* :: *AE* : *ED* and the angles at *C* and *D* are right angles, ∴ the triangles *ABC*, *ADE* are equiangular (Eucl. vi. 7.), and the angle $CAB = DAE$; to each of these add the angle BAD; and $CAD = BAE$.

In the same manner FAG may be shewn to be equal to BAD.

* * *

(26.) *To determine the Arithmetic, Geometric and Harmonic means between two given straight lines.*



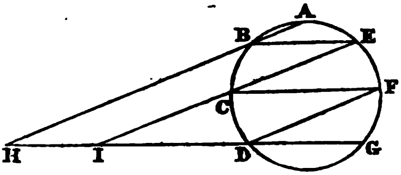
Let AB, BC be the two given lines. Let them be placed in the same straight line, and on AC describe a semicircle ADC. Through B draw BD at right angles to AC, join OD, and upon it let fall the perpendicular BE. Then AO being half of the sum of AB, BC is the arithmetic mean; and since (Eucl. vi. 8.) $AB : BD :: BD : BC$, $\therefore$ BD is the geometric mean. And DE is the harmonic mean, for (Eucl. vi, 8.) $(DO =) AO : DB :: DB : DE$, *i.e.* it is a third proportional to the arithmetic and geometric means, and $\therefore$ is the harmonic mean.

* * *

(27.) *If on each side of any point in a circle any number of equal arcs be taken, and the extremities of each pair joined; the sum of the chords so drawn, will be equal to the last chord produced to meet a line drawn from the given point through the extremity of the first arc.*

Let AB, BC, CD, &c., AE, EF, FG, &c. be equal arcs and let their extremities BE, CF, DG be joined;

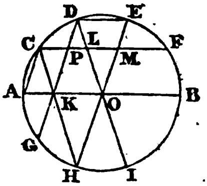
from *A* through *B* draw *ABH* meeting the last chord *GD* in *H*; *GH* is equal to *EB*, *CF*, *DG*, &c. together.



Join *DF*, *EC*, and produce *EC* to *I*. Since *AE* = *BC*, *ABH* is parallel to *ECI*. And since the arcs are equal, the lines *BE*, *CF*, *DG* are parallel, whence *BI* is a parallelogram and *BE* = *HI*. In the same manner it may be shewn, that *CF* = *ID*, and so on, whatever be the number of equal arcs; hence *GH* is equal to the sum of *BE*, *CF*, *DG*.

* * *

(28.) *If the circumference of a semicircle be divided into an odd number of equal parts, and through the points which are equally distant from the diameter, lines be drawn; the segments of these lines intercepted between radii drawn to the extremities of the most remote, will together be equal to a radius of the circle.*



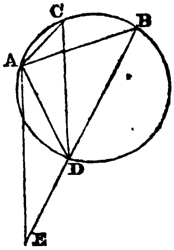
Let the circumference of the semicircle *ADB* be divided into any odd number of equal parts, *e. g.* five, (the demonstration being the same for any odd number) in the points *C*, *D*, *E*, *F*. Join *DE*, *CF*, which are parallel, since they intercept equal arcs. Join *OD*, *OE*; *DE* and *LM* together are equal to the radius of the circle.

For complete the circle, and divide the opposite semicircle in the same manner; join AC, DG, EH which will be parallel to one another; CH will also be parallel to DI. Hence $DE = OK$, and OK is also equal to each of the two PM, CL, $\therefore PM = CL$, whence $LM = CP$ which is equal to AK, since CF is parallel to AB,

and $DE + LM = KO + AK = AO$ the radius of the circle.

* * *

(29.) *If from the extremities and the point of bisection of any arc of a circle, lines be drawn to any point in the opposite circumference; the sum of those drawn from the extremities will have to that from the point of bisection, the same ratio that the line joining the extremities, has to that joining one of them and the point of bisection.*



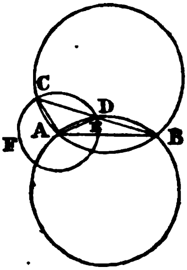
Let the arc AB be bisected in C, and AB, AC joined; to any point D in the circumference draw AD, BD, CD; then $AD + DB : DC :: AB : AC$.

Draw AE parallel to CD, and let it meet BD produced in E. The angle $EAD = ADC = CAB$ (Eucl. iii. 27.); to each of these add DAB, and $EAB = CAD$; also $ABD = ACD$, $\therefore$ the triangles EAB, ACD are equiangular, whence $BE : CD :: BA : AC$. But since CDB is equal to each of the angles AED, DAE, they are equal to one another, $\therefore DA = DE$,

and $AD + DB : DC :: AB : AC$.

* * *

(30.) *If two equal circles cut each other, and from either point of intersection a circle be described cutting them; the points where this circle cuts them and the other point of intersection of the equal circles are in the same straight line.*

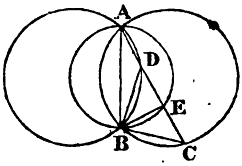


Let the two equal circles cut each other in *A* and *B*, and with the centre *A* and any distance *AC*, describe a circle *FCD* cutting their circumferences in *C* and *D*; *C*, *D*, *B* will be in a straight line.

Join *CB*, and let it meet the circumference *ADB* in *E*. Join *AE*, *AC*. Since the angle *ABC* is an angle in each of the two equal circles, the circumference *AC* is equal to the circumference *AE* (Eucl. iii. 26.), ∴ the line *AC* is equal to the line *AE*; and ∴ *E* is a point in the circle *FDC*, and being by construction in the circumference *ADB*, it must coincide with *D*; ∴ *CB* passes through *D*, or *C*, *D*, *B* are in a straight line.

* * *

(31.) *If two equal circles cut each other, and from either point of intersection a line be drawn meeting the circumferences; the part of it intercepted between the circumferences will be bisected by the circle whose diameter is the common chord of the equal circles.*



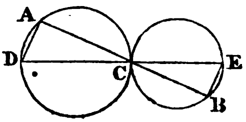
Let the two equal circles *ADB*, *ACB* cut each other in *A* and *B*; join *AB*, and on it as a diameter let a circle *AEB* be described, and from *A* draw any line *ADC* meeting the circumferences in *D* and *C*; *DC* is bisected in *E*.

Join BD, BE, BC. Since the angle CAB is in each of the two equal circles, the circumferences BD, BC on which it stands are equal, and ∴ the straight lines BD, BC are equal, and consequently the angle BDE is equal to the angle BCE; and the angle BED in a semicircle is a right angle, and ∴ equal to BEC, and BE is common to the two triangles BED, BEC, ∴ $DE = EC$.

* * *

(32.) *If two circles touch each other externally or internally; any straight line drawn through the point of contact, will cut off similar segments.*

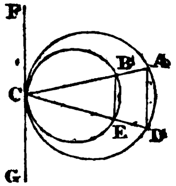
Let the circles ADC, BCE touch each other in the point C, and let any line ACB be drawn through the point of contact; it will cut off similar segments.



For draw the diameters CD, CE, and join AD, BE. Then DCE being a straight line (Eucl. iii. 12.), the angle ACD is equal to BCE, and $DAC = CBE$ each being in a semicircle, and ∴ a right angle, whence the angles ADC, CEB are equal, and the segments ADC, CEB similar; and ∴ the segments AC and CB are also similar.

* * *

(33.) *If two circles touch each other externally or internally; two straight lines drawn through the point of contact will intercept arcs, the chords of which are parallel.*



Let the two circles *ACD*, *ECB* touch each other in *C*, and let *ABC*, *DEC* be any two lines drawn through the point of contact. Draw the tangent *FCG*, and join *AD*, *EB*.

Then (Eucl. iii. 32.) the angle *ADC* = (*FCA* =) *BEC*, whence (Eucl. i. 28.) *AD* is parallel to *BE*.

* * *

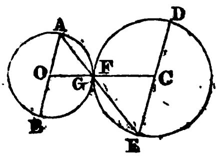
(34.) *If two circles touch each other internally or externally, any two straight lines drawn through the point of contact and terminated both ways by the circumferences will be cut proportionally by the circumference.*

Let the two circles touch each other in *C*, (see last Fig.) and let *ACB*, *DCE* be any two lines drawn through the point of contact; then it may be shewn (as in the last prop.), that *AD* is parallel to *BE*, and the triangles *ACD*, *BCE* are similar,

$$\therefore AC : CB :: DC : CE.$$

* * *

(35.) *If two circles touch each other externally, and parallel diameters be drawn, the straight line joining the extremities of these diameters, will pass through the point of contact.*



Let *ABG*, *DGE* be two circles touching each other externally in the point *G*; and let *AB*, *DE* be parallel diameters, join *AE*; *AE* will pass through *G*.

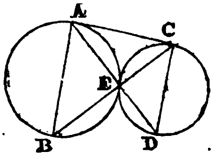
Join O, C the centres of the circles, OC will pass through G: let it meet AE in F. The vertically opposite angles at F being equal, and also the alternate angles OAF, FEC, the triangles AOF, FCE are equiangular,

$$\therefore AO : CE :: OF : FC,$$

$$\textit{comp.}\quad AO + CE : CE :: OF + FC : FC.$$

But $OC = AO + CE$, and $\therefore FC = CE = CG$, and consequently F and G coincide, or AE intersects OC in the point G, *i. e.* it passes through the point of contact.

(36.) *If two circles touch each other, and also touch a straight line; the part of the line between the points of contact, is a mean proportional between the diameters of the circle.*

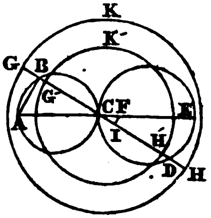


Let AEB, CED be two circles touching each other in E, and a straight line AC in A and C; draw the diameters AB, CD; AC is a mean proportional between AB and CD.

Join AD, BC; these lines (ii. 25.), pass through the point of contact C. And since CA touches the circle in A, from which AE is drawn, the angle CAD is equal to the angle in the alternate segment ABE; also the angle ACD being a right angle is equal to the angle CAB, $\therefore$ the triangles ACD, ABC are equiangular, and

$$BA : AC :: AC : CD.$$

(37.) *If two circles touch each other externally, and the line joining their centres be produced to their circumferences; and from its middle point as a centre with* *any radius whatever a circle be described, and any line placed in it passing through the point of contact; the parts of the line intercepted between the circumference of this circle and each of the others will be equal.*



Let ABC, DCE be two circles which touch each other externally in C, and let AFE be the line joining their centres and produced to the circumferences in A and E. Bisect AE in F, and with the centre F and any radius, let a circle GHK be described, and in it any line GCH drawn through C meeting the circumferences of the circles in B and D; then will $GB = DH$.

Join AB, DE, and draw FI parallel to AB; it will be perpendicular to GH since ABC is an angle in a semicircle, and $\therefore$ GH is bisected in I. And since IF is parallel to AB,

$$\text{(Eucl. vi. 2.)}\ AF : BI :: FC : IC,$$

also the triangles ICF, ECD being similar,

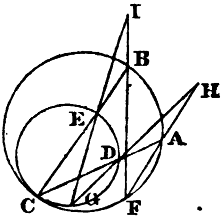
$$FC : CI :: EF : ID,$$

$$\therefore \text{(Eucl. v. 15.)}\ AF : BI :: EF : ID.$$

$$\text{But } AF = FE, \ \therefore BI = ID,$$

and it has been shewn that $GI = IH$, whence $GB = DH$.

(38.) *If from the point of contact of two circles which touch each other internally any number of lines be drawn; and through the points, where these intersect the circumferences, lines be drawn from any other point in each circumference, and produced to meet; the angles formed by these lines will be equal.*

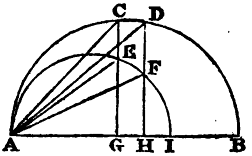


Let the two circles ABC, DEC touch each other internally in C, from which let any lines CA, CB be drawn; and taking any two points G and F, through E and B draw GEI, FBI, and through D and A draw GDH, FAH; if those lines meet, the angle at I will be equal to the angle at H.

For the angles CBF, CAF standing on the same circumference CF, are equal, $\therefore$ the angle IBE is equal to HAD. Also the angles CEG, CDG, standing on the same circumference CG, are equal, and $\therefore$ the angle IEB is equal to the angle HDA; $\therefore$ the triangles IEB, HDA have two angles in each equal, and consequently the remaining angles are equal, *i. e.* $EIB = DHA$.

* * *

(39.) *If two circles touch each other internally, and any two perpendiculars to their common diameter be produced to cut the circumference; the lines joining the points of intersection and the point of contact are proportionals.*



Let the two circles ACB, AEI touch each other internally in the point A, from which let the common diameter AIB be drawn, and from any two points G, H let perpendiculars GC, HD meet the circumferences in C, D, E, F; join AC, AD, AE, AF; these lines are proportional.

For since $AB : AD :: AD : AH$,

$AB : AH$ in the duplicate ratio of $AB : AD$.

For the same reason,

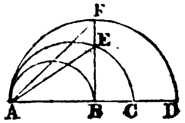
$AG : AB$ in the duplicate ratio of $AC : AB$,

$\therefore AG : AH$ in the duplicate ratio of $AC : AD$.
In the same manner it may be shewn that

$AG : AH$ in the duplicate ratio of $AE : AF$,
$\therefore$ (Eucl. v. 15.) the duplicate ratio of $AC : AD$,
is the same with the duplicate ratio of $AE : AF$,
and $\therefore AC : AD :: AE : AF$.

* * *

(40.) *If three circles, whose diameters are in continued proportion touch each other internally, and from the extremity of the least diameter passing through the point of contact, a perpendicular be drawn, meeting the circumferences of the other two circles; this diameter and the lines joining the points of intersection and contact are in continued proportion.*



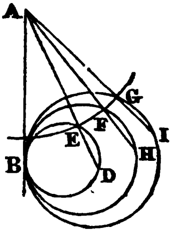
Let AB, AC, AD the diameters of three circles touching each other in A, be in continued proportion, viz. $AB : AC :: AC : AD$, and from B the perpendicular BF meet the circumferences in E and F; join AE, AF; then $AB : AE :: AE : AF$.

For (Eucl. vi. 8.) $AB : AF :: AF : AD$.
But by the hypothesis $AC : AB :: AD : AC$,
$\therefore AC : AF :: AF : AC$,
whence $AF = AC$.

And (Eucl. vi. 8.) $AB : AE :: AE : AC$,
$\therefore AB : AE :: AE : AF$.

* * *

(41.) *If a common tangent be drawn to any number of circles which touch each other internally; and from* *any point in this tangent as a centre, a circle be described cutting the others, and from this centre lines be drawn through the intersections of the circles respectively; the segments of them within each circle will be equal.*



Let the circles touch each other in the point B, to which let a tangent BA be drawn, and from any point A in it as a centre with any radius, let a circle EFG be described. Draw the lines AED, AFH, AGI; then will the parts DE, HF, IG be equal.

For since AB touches the circle, (Eucl. iii. 36.)

$$DA : AB :: AB : AE,$$

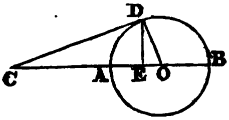
For the same reason, $AB : AH :: AF : AB$,

$\therefore$ *ex æquo* $DA : AH :: AF : AE$,

but $AF = AE$, $\therefore DA = AH$ and consequently $DE = HF$. In the same manner it may be proved, that $IG = HF$ or DE.

* * *

(42.) *If from any point in the diameter of a circle produced, a tangent be drawn; a perpendicular from the point of contact to the diameter will divide it into segments which have the same ratio that the distances of the point without the circle from each extremity of the diameter, have to each other.*



From any point C in the diameter BA produced, let a tangent CD be drawn, and from D, draw DE perpendicular to AB; $AE : EB :: AC : CB$.

Take O the centre of the circle, join DO; then (Eucl. iii. 18.) the angle CDO is a right angle, and $\therefore$ (Eucl. vi. 8.)

$$CO : OD :: OD : OE,$$

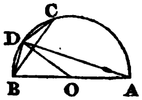
$$\text{or } CO : OA :: OA : OE,$$

$$\therefore \textit{div.} \text{ and } \textit{comp.}\ AC : CB :: AE : EB.$$

Cor. The converse may easily be proved.

* * *

(43.) *If from the extremity of the diameter of a given semicircle a straight line be drawn in it, equal to the radius, and from the centre a perpendicular let fall upon it and produced to the circumference; it will be a mean proportional between the lines drawn from the point of intersection with the circumference to the extremities of the diameter.*



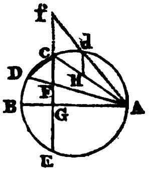
From B the extremity of the diameter AB let a line BC be drawn, equal to the radius BO; and on it let fall a perpendicular OD meeting the circumference in D; join DB, DA; DO is a mean proportional between DA and DB.

Join DC. Then the angles BAD, BCD on the same base are equal. Also since OD bisects BC, it bisects the arc BDC, $\therefore$ also the straight line $BD = DC$ and the angle $DBC = DCB$, but $ODA = OAD$, $\therefore$ the triangles ODA, DBC are similar, $\therefore AD : DO :: (BC =)\, DO : DB$.

* * *

(44.) *If from the extremity of the diameter of a circle, two lines be drawn, one of which cuts a perpen-*

dicular to the diameter, and the other is drawn to the point where the perpendicular meets the circumference; the latter of these lines is a mean proportional between the cutting line, and that part of it which is intercepted between the perpendicular and the extremity of the diameter.



Let CE be at right angles to the diameter AB of the circle ABC, and from A let AD, AC be drawn, of which AD cuts CE in F, then will

$$AD : AC :: AC : AF.$$

For since the circumference AE is equal to the circumference AC, (Eucl. iii. 27.) the angle ECA is equal to the angle ADC, and the angle at A is common to the two triangles ADC, ACF, $\therefore$ the triangles are similar, and

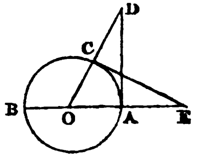
$$AD : AC :: AC : AF.$$

But if the point of intersection f be without the circle, draw dH parallel to CG, then, as before, the angle HdA is equal to ACd, and the angle at A common to the triangles AHd, ACd,

$$\therefore Ad : AC :: AH : Ad :: AC : Af.$$

* * *

(45.) *In the diameter of a circle produced, to determine a point, from which a tangent drawn to the circumference shall be equal to the diameter.*

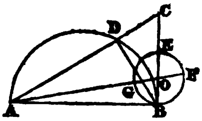


From A the extremity of the diameter AB, draw AD at right angles and equal to AB. Find the centre O, join OD cutting the circle in C; and through C draw CE at right angles to OD meeting BA produced in E.

Then because the angle OAD is equal to OCE, each being a right angle, and the angle at O is common to the two triangles OAD, OCE, and $OA = OC$, $\therefore AD = CE$. But AD was made equal to AB, $\therefore CE = AB$, and E is the point required.

* * *

(46.) *To determine a point in the perpendicular at the extremity of the diameter of a semicircle, from which if a line be drawn to the other extremity of the diameter, the part without the circle may be equal to a given straight line.*



From B the extremity of the diameter of the semicircle ADB, let a perpendicular BC be drawn; in which take BE equal to the given line; and on it as a diameter describe a circle; through the centre of which draw AGF, and with A as centre and radius AF describe a circle cutting BC in C. Join AC; CD is equal to the given line.

Join BD. Then BD being perpendicular to AC,

(Eucl. vi. 8. Cor.) $AC : AB :: AB : AD$,

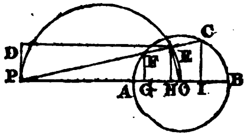
and (Eucl. iii. 36.) $AB : AF :: AG : AB$,

$\therefore$ *ex æquo*, $AC : AF :: AG : AD$,

whence $AG = AD$, and $\therefore DC = GF = BE$.

* * *

(47.) *Through a given point without a given circle, to draw a straight line to cut the circle, so that the two perpendiculars drawn from the points of intersection to that diameter which passes through the given point, may* *together be equal to a given line, not greater than the diameter of the circle.*

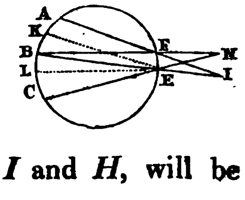


Let *P* be the given point without the circle *ABC*, whose centre is *O*; *AB* the diameter which passes through *P*. On *PO* describe a semicircle. From *P* draw *PD* at right angles to *PB* and equal to half the given line; through *D* draw *DE* parallel to *PB* meeting the semicircle in *E*; join *PE*; and produce it to *C*; *PC* is the line required.

For, draw *FG*, *EH*, *CI* perpendiculars to *AB*. Join *OE*; then the angle *PEO* is a right angle, and ∴ (Eucl. iii. 3.) $EF = EC$; whence *FG* and *CI* together are equal to $2EH = 2PD =$ the given line.

* * *

(48.) *If from each extremity of any number of equal adjacent arcs in the circumference of a circle, lines be drawn through two given points in the opposite circumference, and produced till they meet; the angles formed by these lines will be equal.*

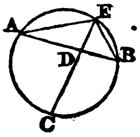


Let *AB*, *BC*, be equal arcs, and *F*, *E* two points in the opposite circumference, through which let the lines *AFI*, *BEI*; *BFH*, *CEH* be drawn, so as to meet; the angles at *I* and *H*, will be equal.

From *E* draw *EK*, *EL*, respectively parallel to *FA*, *FB*. Since *EK* is parallel to *FA*, the angle *KEB* is equal to the angle at *I*; for the same reason the angle *LEC* is equal to the angle at *H*. But since the arcs *AB*, *BC*, are equal, and *AK*, *BL* being each equal to *EF* (ii. 1.) are also equal to one another, ∴ *KB*, *LC*, are also equal, and (Eucl. iii. 27.) the angles *KEB*, *LEC*, are equal, ∴ also the angles at *I* and *H* are equal. The same may be proved whatever be the number of equal arcs *AB*, *BC*.

* * *

(49.) *To determine a point in the circumference of a circle, from which lines drawn to two other given points, shall have a given ratio.*



Let *A*, *B* be the two given points, join *AB*, and divide it in *D* so that *AD* : *DB* may be in the given ratio; bisect the arc *ACB* in *C*, join *CD*, and produce it to *E*; *E* is the point required.

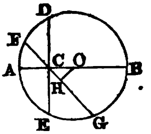
Join *AE*, *EB*. Since *AC*=*CB*, the angle *AEC* is equal to the angle *CEB*, ∴ *AB* is cut by the line *ED* bisecting the angle *AEB*, and consequently (Eucl. vi. 3.) *AE* : *EB* :: *AD* : *DB*, *i. e.* in the given ratio.

* * *

(50.) *If any point be taken in the diameter of a circle, which is not the centre; of all the chords which can be drawn through that point, that is the least which is at right angles to the diameter.*

In *AB* the diameter of the circle *ADB*, let any point

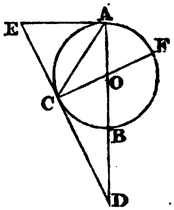
C be taken which is not the centre, and let *DE, FG* be any chords drawn through it, of which *DE* is perpendicular to *AB*; *DE* is less than *FG*.



Take *O* the centre and draw *OH* perpendicular to *FG*. Now in the triangle *OCH*, the angle at *H* is a right angle and ∴ greater than the angle *OCH*, ∴ *CO* is greater than *OH* and consequently (Eucl. iii. 15.) *DE* is less than *FG*.

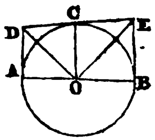
(51.) *If from any point without a circle lines be drawn touching it; the angle contained by the tangents, is double the angle contained by a line joining the points of contact and the diameter drawn through one of them.*

From the point *E* without the circle *ABC* let *EA, ECD* be drawn touching the circle in *A* and *C*, and let *ED* meet the diameter *AB*, drawn from *A*, in the point *D*. Join *AC*; the angle *AEC* is double of *CAB*.



Through *C* draw the diameter *COF*, then the angle *FCD* is a right angle and ∴ equal to *EAD*, and *EDA* is common to the triangles *EDA*, *COD*, ∴ the angle *COD* is equal to *AED*. But *COB* is double of *CAD*, ∴ *AEC* is double of *CAD*.

(52.) *If from the extremities of the diameter of a circle tangents be drawn, and produced to intersect a tangent to any point of the circumference, the straight lines joining the points of intersection, and the centre of the circle form a right angle.*



From A and B the extremities of the diameter AB let tangents AD, BE be drawn, meeting a tangent to any other point C of the circumference, in D and E; and let O be the centre; join DO, EO; the angle DOE is a right angle.

Join CO. Then since $CE = EB$, $CO = OB$, and the angles at C and B, being right angles, are equal, ∴ the angle $CEO = OEB$ and CEB is bisected by EO. In the same manner it may be shewn that the angle ADC is bisected by DO. And since the angles CEB, CDA are equal to two right angles, ∴ CDO and CEO are equal to one right angle, and ∴ (Eucl. i. 32.) DOE is a right angle.

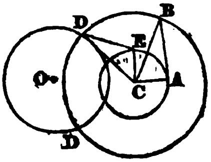
(53.) *If from the extremities of the diameter of a circle tangents be drawn; any other tangent to the circle, terminated by them, is so divided at the point of contact, that the radius of the circle is a mean proportional between its segments.*

Let AD, BE be two lines touching the circle ABC, (see the last Fig.) at A and B the extremities of its diameter, and meeting DCE any other tangent to the circle; take O the centre, and join CO; then will $DC : CO :: CO : CE$.

Join DO, EO; then as in the last proposition, it may be shewn that DOE is a right angle; and since from the right angle OC is drawn perpendicular to the base, ∴ (Eucl. vi. 8.) it is a mean proportional between the segments of the base, or

$$DC : CO :: CO : CE.$$

(54.) *Two circles being given in magnitude and position; to find a point in the circumference of one of them, to which if a tangent be drawn cutting the circumference of the other, the part of it intercepted between the two circumferences may be equal to a given line.*



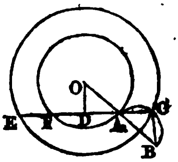
Let *O* and *C* be the centres of the two given circles. To any point *A* in the circumference of one of them let a tangent *AB* be drawn, and make *AB* equal to the given line. With the centre *C* and distance *CB* describe a circle *DBD* cutting the other in the point *D*, and from *D* draw *DE* touching the former given circle; *E* will be the point required.

Join *CA*, *CB*, *CD*, *CE*. Since *CA* = *CE* and *CB* = *CD*, and the angles at *A* and *E* are right angles, ∴ *DE* is equal to *BA*, *i. e.* to the given line.

If the circle *DBD* neither cuts nor touches *DD*, it is evident the problem will be impossible.

* * *

(55.) *To draw a straight line cutting two concentric circles so that the part of it which is intercepted by the circumference of the greater may be double the part intercepted by the circumference of the less.*



Let *O* be the centre of the two circles. Draw any radius *OA* of the lesser circle and produce it to *B*, making *AB* = *AO*. On *AB* describe a semicircle *ACB* cutting the greater circumference in *C*; join *AC*, and produce it to *E*; *CE* is the line required.

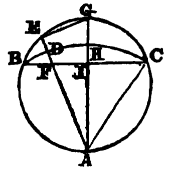
Join *CB*; and let fall the perpendicular *OD*. Then the angle ADO being a right angle is equal to the angle ACB, and the vertically opposite angles at A are equal, and the side $OA=AB$, $\therefore AC=AD$, and $DC=2AD$; but DC is half of EC and AD half of AF, $\therefore EC$ is double of AF.

Cor. The same construction will apply whatever be the relation required between the two chords. Take $OB : OA$ in the required ratio, and proceed as in the proposition.

* * *

(56.) *If two circles intersect each other, the centre of the one being in the circumference of the other, and any line be drawn from that centre; the parts of it which are cut off by the common chord, and the two circumferences will be in continued proportion.*

From any point A in the circumference of the circle ABG, as a centre and with any radius, let a circle BDC be described, cutting the former in B and C. Join BC, and from A draw any line AFE, $AF : AD :: AD : AE$.



From A draw the diameter AG, it will cut BC at right angles in I. Join GE, AC. The right angle AIF being equal to the right angle AEG, and the angle at A common, the triangles AIF, AEG are similar,

$$\therefore AF : AI :: AG : AE.$$

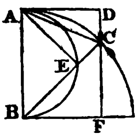
But (Eucl. vi. 8. Cor.) $AI : AC :: AC : AG$,

$$\therefore \textit{ex æquo},\ AF : AC :: AC : AE,$$

$$\text{or}\ AF : AD :: AD : AE.$$

* * *

(57.) *If a semicircle be described on the side of a quadrant, and from any point in the quadrantal arc a radius be drawn, the part of this radius intercepted between the quadrant and semicircle, is equal to the perpendicular let fall from the same point on their common tangent.*



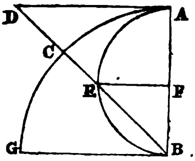
On AB the side of a quadrant let the semicircle AEB be described, and from any point C draw the radius CB, and CD perpendicular to AD a tangent at A; $EC = CD$.

Join AE, AC; then the angle AEB being in a semicircle, its adjacent angle AEC is a right angle and $\therefore$ equal to ADC; and $BCA = BAC = ACD$ the alternate angle; $\therefore$ the two triangles AEC, ACD have two angles in each equal, and one side AC common, $\therefore$ $EC = CD$.

Cor. Any chord of the semicircle drawn from the centre of the quadrant, is equal to the perpendicular drawn to the other side from the point in which the chord produced meets the quadrantal arc.

Produce DC to F; then CE being equal to CD, the remainder BE is equal to the remainder CF.

(58.) *If a semicircle be described on the side of a quadrant, and a line be drawn from the centre of the quadrant to a common tangent; this line, the parts of it cut off by the circumferences of the quadrant, and of the semicircle, and the segment of the diameter of the semicircle made by a perpendicular from the point where the line meets its circumference, are in continued proportion.*



On the radius AB of the quadrant AGB let the semicircle AEB be described, and at A draw the tangent AD. From B draw any line $BECD$ meeting the tangent in D, and the circumferences in C, E; from E let fall the perpendicular EF; then BD, BC, BE, BF are in continued proportion.

Since FE is perpendicular to BA, it is parallel to AD,

$$\therefore BF : BE :: (BA=)\ BC : BD,$$

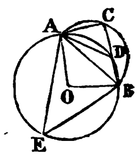
But (Eucl. vi. 8.) $BF : BE :: BE : (BA=)\ BC$,

$\therefore$ (Eucl. v. 15.) also $BE : BC :: BC : BD$,

and $BF : BE :: BE : BC :: BC : BD$.

* * *

(59.) *If the chord of a quadrant be made the diameter of a semicircle, and from its extremities two straight lines be drawn to any point in the circumference of the semicircle; the segment of the greater line intercepted between the two circumferences shall be equal to the less of the two lines.*

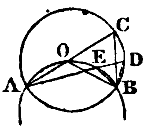


Let O be the centre of the quadrant ADB, join AB and on it let a semicircle ACB be described; from any point C in which let lines CA, CB be drawn to A and B, of which CB is the greater; $CD=CA$.

Join AD, and complete the circle ABE; take any point E and join EA, EB. Since $ADBE$ is a quadrilateral figure inscribed in a circle, the angles AEB, ADB are equal to two right angles, and $\therefore$ equal to ADB, ADC; whence $AEB=ADC$; but AEB is half of AOB which is a right angle, $\therefore ADC$ is half a right angle, and ACD being a right angle (Eucl. ii. 31.), CAD is half a right angle, and $\therefore$ equal to CDA, consequently $CA = CD$.

* * *

(60.) *If two circles cut each other so that the circumference of one passes through the centre of the other, and from either point of intersection a straight line be drawn cutting both circumferences; the part intercepted between the two circumferences will be equal to the chord drawn from the other point of intersection to the point where it meets the inner circumference.*



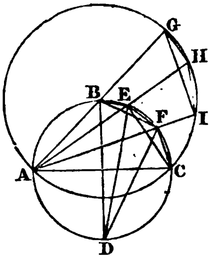
Through O the centre of the circle ABC, let the circle AOB be described, cutting ABC in A and B. If any line AED be drawn from A, and BE joined; DE will be equal to EB.

Draw the diameter AOC; join BC, BD. Then since the angle AOB is equal to AEB, $\therefore$ the angle COB is equal to DEB. Also the angles OCB, EDB, being in the same segment, are equal to one another, $\therefore$ the triangles OCB, EDB are equiangular, and $\therefore$ since $OB = OC$, the angle OCB is equal to the angle OBC, whence $EDB = EBD$, and $\therefore$ $ED = EB$.

* * *

(61.) *If from each extremity of the diameter of a circle lines be drawn to any two points in the circumference; the sums of the lines so drawn to each point will have to one another the same ratio that the lines have, which join those points and the opposite extremity of a diameter perpendicular to the former.*

From A and C the extremities of AC the diameter of the circle ABC, let lines AE, EC, AF, FC be drawn to any points E and F in the circumference, and draw the diameter BD perpendicular to AC; join ED, FD; then



$$AE + EC : AF + FC :: ED : FD.$$

Join AB; and with the centre B and distance BA describe a circle AGC; produce AB, AE, AF to the circumference. Join GH, HI, BE, EF, GI, BF. Then since AG and BD are diameters of the circles, the angles AHG, AIG are equal to DEB, DFB; but BAE, EAF are equal to BDE, EDF, and the angle HIG being $= HAG = BDE = BFE$, $\therefore$ the angle $HIA = EFD$ and the triangles GAH, HAI are similar to BDE, EDF, and

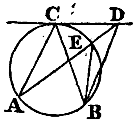
$$\therefore AH : AI :: ED : FD,$$

But (ii. 60.) $EH = EC$, and $FI = FC$,

$$\therefore AE + EC : AF + FC :: ED : FD.$$

* * *

(62.) *If from any two points in the circumference of a circle there be drawn two straight lines to a point in a tangent to that circle; they will make the greatest angle when drawn to the point of contact.*



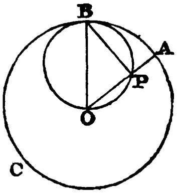
Let A and B be the two points, and CD the tangent at C; join AC, CB; the angle ACB is greater than any other angle ADB formed by lines drawn to any other point D.

Join BE. Then the angles ACB, AEB in the same segment are equal; but ADB is less than the exterior angle AEB, and ∴ is less than ACB.

Cor. If two circles touch each other in C, it might be shewn in a similar manner, that the angle formed by two straight lines drawn from A and B to C the point of contact; will be greater than the angle formed by lines drawn from the same points to any point in the exterior circle.

* * *

(63.) *From a given point within a given circle to draw a straight line which shall make with the circumference an angle less than the angle made by any other line drawn from that point.*



Let P be the given point within the circle ABC.

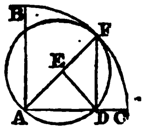
Find O the centre, join OP and produce it to the circumference. From P draw PB at right angles to OA; it is the line required.

Join OB, and on it as a diameter describe a circle OPB, which will touch the circle ABC in B. Then OBP is the greatest angle that can be included between lines drawn from O and P to the circumference ABC (ii. 63. Cor.), ∴ the angle contained by PB and the circumference AB will be the least.

* * *

(64.) *To determine a point in the arc of a quadrant, from which if lines be drawn to the centre and the point of bisection of the radius, they shall contain the greatest possible angle.*

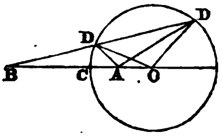
Let BC be the arch of a quadrant whose centre is A and let the radius AC be bisected in D. On AD describe an equilateral triangle ADE, and produce AE to F; F is the point required.



Join FD. Then $AF=AC$, and $AD=AE$, but AD is half of AC, and $\therefore$ AE is half of AF, and $\therefore$ equal to EF; and EA, ED, EF are equal; whence a circle described from the centre E at the distance of any one of them will pass through the extremities of the other two, and touch the arc BC in F, because their centres are in the same straight line; and AFD (ii. 63. Cor.) is greater than any other angle formed by lines drawn from any point in BC to A and D.

(65.) *If the radius of a circle be a mean proportional to two distances from the centre in the same straight line; the lines drawn from their extremities to any point in the circumference will have the same ratio that the distances of these points from the circumference have.*

Let A and B be the two points, such that $AO : CO :: CO : BO$, and from A and B let lines AD, BD be drawn to any point D in the circumference; these have always the same ratio, viz. $AD : BD :: AC : BC$.



Join OD. Then since $OD=OC$, $OA : OD :: OD : OC$, *i. e.* the sides about the common angle AOD of the triangles AOD, BOD are proportional, and $\therefore$ the triangles are similar, consequently

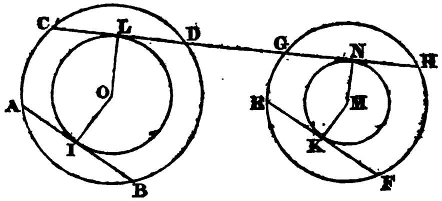
$$DA : DB :: OA : (OD=)\,OC.$$

But since $AO : CO :: CO : BO$,
div. $AO : AC :: CO : CB$,
alt. $AO : CO :: AC : CB$,
$\therefore$ (Eucl. v. 15.) $DA : DB :: AC : CB$.

* * *

(66.) *Two circles being given in position and magnitude; to draw a straight line cutting them so that the chords in each circle may be equal to a given line, not greater than the diameter of the smaller circle.*

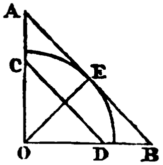
Let *ABC*, *EFG* be the given circles whose centres are *O* and *M*. In each place a line *AB*, *EF*, equal to the given line; and from the centres draw the perpendiculars *OI*, *MK*; and with these distances and centres *O* and *I* describe circles which will touch *AB*, *EF* in *I* and *K*; draw *CDGH* (ii. 7.) which shall touch these circles in *L* and *N*; each of the chords *CD* and *GH* will be equal to the given line.



Join *OL*, *MN*; these lines are perpendicular to *CD* and *GH*, and being respectively equal to *OI* and *MK*, $CD = AB$ (Eucl. iii. 14.) and $GH = EF$; but *AB* and *EF* are each equal to the given line, $\therefore$ *CD* and *GH* are also each equal to the given line.

Cor. If the intercepted parts are required to have a given ratio, take AB and EF in that ratio, and make the same construction as in the proposition.

(67.) *To determine a point in the arc of a quadrant, through which if a tangent be drawn meeting the sides of the quadrant produced, the intercepted parts may have a given ratio.*



Let OA, OB be the sides of a quadrant produced; and take M and N two right lines which are in the given ratio, and let OC be a mean proportional between the radius of the quadrant and M, and OD a mean proportional between the radius and N. Join CD, and draw the radius OE cutting it at right angles; E is the point required.

Through E draw the tangent AEB, which being perpendicular to OE (Eucl. iii. 18.), will be parallel to CD,

$$\therefore AO : OB :: CO : OD,$$

and since OC and OD are mean proportionals between M and the radius, and N and the radius respectively,

$M : N$ in the duplicate ratio of $OC : OD$,

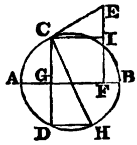
i. e. in the duplicate ratio of $AO : OB$.

But (Eucl. vi. 8. Cor.)

$AE : EB$ in the duplicate ratio of $AO : OB$,

$\therefore AE : EB :: M : N$, *i. e.* in the given ratio.

(68.) *If a tangent be drawn to a circle at the extremity of a chord which cuts the diameter at right angles, and from any point in it a perpendicular be let fall; the* *segment of the diameter intercepted between that perpendicular and chord is to the intercepted part of the tangent, as the chord is to the diameter.*



Let the chord CD be perpendicular to the diameter AB, and let CE touch the circle at C; from any point E in which let EF be drawn perpendicular to AB;

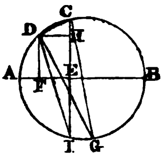
$$FG : CE :: CD : AB.$$

Draw the diameter CH; join HD and draw CI perpendicular to EF. Since EC touches the circle, the angle ECH (Eucl. iii. 18.) is a right angle and ∴ equal to ICD; whence, taking away from each ICH, the angle $ECI = HCD$, and EIC, HDC are right angles, ∴ the triangles ECI, HDC are equiangular,

$$\text{whence } IC : CE :: DC : CH,$$

$$\text{or } GF : CE :: CD : AB.$$

(69.) *If a straight line be placed in a circle, and from its extremities perpendiculars be let fall upon any diameter; these perpendiculars together will have to the part of the diameter intercepted between them, the same ratio that a line placed in the circle perpendicular to the former line, has to the former line itself.*



Let the line CD be placed in the circle ABC, and from its extremities let CE, DF be drawn perpendicular to a diameter AB. From D let DG be drawn perpendicular to DC; then will

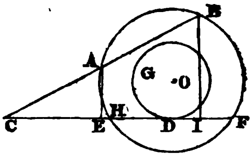
$$CE + DF : EF :: GD : DC.$$

Join CG, which is therefore a diameter of the circle;

and produce CE to I: join DI, and draw DH perpendicular to CE. Since CI is perpendicular to AB, $CE = EI$, but $HE = DF$, $\therefore HI = CE + DF$. Now (Eucl. iii. 21.) the angle at G is equal to the angle at I, and CDG, DHI are right angles, $\therefore$ the triangles CGD, HID are equiangular,

$$\text{and } HI : HD :: DG : DC,$$
$$\text{or } CE+DF : EF :: DG : DC.$$

(70.) *In a circle to place a straight line of given length, so that perpendiculars drawn to it from two given points in the circumference may have a given ratio.*

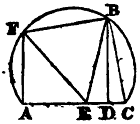


Let A and B be the given points in the circumference of the circle whose centre is O. Join BA and produce it, and take $AC : CB$ in the given ratio. In the circle place a straight line equal to the given straight line, and from the centre O let fall a perpendicular upon it. With O as centre, and distance equal to this perpendicular describe a circle DG, and from C draw $CEDF$ a tangent to it; then HF is the line required.

For (Eucl. iii. 14.) it is equal to the given straight line. And if from A and B, AE, BI be drawn perpendicular to CF, they are parallel to each other, and the triangles CAE, CBI are similar,

$$\therefore AE : BI :: CA : CB, \text{ i.e. in the given ratio.}$$

(71.) *If from any point in the arc of a segment of a circle a line be drawn perpendicular to the base; and* *from the greater segment of the base, and arc, parts be cut off respectively equal to the less; the remaining part of the base shall be equal to the chord of the remaining arc.*

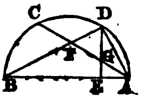


From any point B in the arc ABC, let BD be drawn perpendicular to AC; make $BF = BC$, and $DE = DC$; join AF; AF will be equal to AE.

Join FE, EB, FB, BC. Since the arc $BC =$ the arc BF, the straight line $BC = BF$; and DE being equal to DC, and DB common, and at right angles to EC, $\therefore$ $BE = BC = BF$, and the angle BFE is equal to the angle BEF. Now since $AFBC$ is a quadrilateral figure inscribed in a circle, the angles AFB, ACB are equal to two right angles, and $\therefore$ equal to AEB, CEB, of which $ACB = CEB$, $\therefore AFB = AEB$; but $BFE = BEF$, consequently $AFE = AEF$; whence $AF = AE$.

* * *

(72.) *If from the point of bisection of any arc of a circle a perpendicular be drawn to the diameter, which passes through one extremity; it will bisect the segment of the chord cut off by the line joining the point of bisection of the arc and the other extremity of the diameter.*

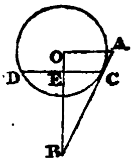


Let AC be the arc bisected in D. Join AC, and from D draw DE perpendicular to the diameter AB and meeting AC in G; join BD; $AG = GF$.

Because AC is bisected in D, the angle CAD is equal to the angle DBA, *i. e.* to the angle EDA (Eucl.

vi. 8.), ∴ the right-angled triangle ADF is equiangular to the two triangles BED, DEA ∴ the angle $GFD = GDF$, and consequently $GD = GF$; also $GAD = GDA$, ∴ $AG = GD$, whence $AG = GF$.

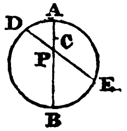
(73.) *In a given circle to draw a chord parallel to a straight line given in position; so that the chord and perpendicular drawn to it from the centre may together be equal to a given line.*



Let O be the centre of the circle, OA the straight line given in position; draw OB perpendicular to it, and equal to the given line. Take OA equal half of OB, and join AB cutting the circle in C; through C draw CD parallel to OA; CD is the chord required.

Because OA is half of OB, and OA, EC are parallel, ∴ (Eucl. vi. 2.) EC is half of EB, and $DC = EB$; therefore DC and OE together are equal to BE and OE together, *i. e.* to BO or to the given line.

(74.) *Through a given point within a given circle, to draw a straight line such that the parts of it intercepted between that point and the circumference may have a given ratio.*



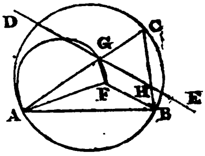
Let P be the given point within the circle ABD. Through P draw the diameter APB, and take $AP : PC$ in the given ratio. With P as centre, and radius equal to a mean proportional between BP and PC

describe a circle cutting ADB in D; join DP and produce it to E; DE is the chord required.

Since $BP : PD :: PD : PC$,
and (Eucl. iii. 35.) $BP : PD :: PE : PA$,
$\therefore PE : PA :: PD : PC$,
and *alt.* $PE : PD :: AP : PC$, *i. e.* in the given ratio.

Cor. Since one circle cuts another in two points, there will be two chords which answer the conditions. If C coincides with A, the ratio is one of equality, and DE will be perpendicular to AB.

(75.) *From two given points in the circumference of a given circle, to draw two lines to a point in the circumference, which shall cut a line given in position, so that the part of it intercepted by them may be equal to a given line.*

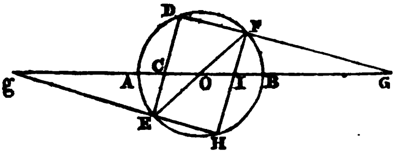


Let A, B be the given points in the circumference of the circle ABC; DE the line given in position. From B draw BF parallel to DE, and equal to the given line. Join AF, and on it describe a segment of a circle AGF containing an angle equal to the angle in the segment ACB; and let it cut DE in G. Join AG, and produce it to C, and join BC cutting DE in H. AC, BC are the lines required.

Join GF. Since the angle $AGF = ACB$, GF is parallel to CB, but FB is parallel to GH, whence $FGHB$ is a parallelogram, and $GH = FB$.

(76.) *If a chord and diameter of a circle intersect each other at any angle, and a perpendicular to the chord be drawn from either extremity of it, meeting the circumference and diameter produced; the whole perpendicular has to the part of it without the circle, the same ratio that the greater segment of the chord has to the less.*

Let the diameter AB and chord DE intersect each other at C; and from D draw DG perpendicular to DE,



meeting AB produced in G;

then $DG : GF :: DC :: CE$.

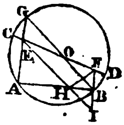
Through F draw FI parallel to DE, and meeting the diameter in I. Join FE, cutting the diameter in O. Since the angle FDE is a right angle, FE is a diameter and O is the centre. And since the angle IFO is equal to the alternate angle OEC, and the angles at O are equal, and $FO = OE$, $\therefore$ the triangles OFI, OEC are equal, and $CE = FI$. And since FI is parallel to DC,

(Eucl. vi. 2.) $GD : GF :: DC : (FI =)\ CE$.

In a similar manner it may be shewn that $gH : gE :: DC : CE$.

* * *

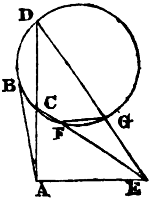
(77.) *If from the extremities of any chord of a circle, perpendiculars to it be drawn and produced to cut a diameter; and from the points of intersection with the diameter lines be drawn to a point in the chord, so as to* *make equal angles with it; these lines together will be equal to the diameter of the circle.*



Let AB be any chord of the circle ABC; draw AE and BF perpendicular to it, meeting the diameter CD in E and F; from which let the lines EH, FH, be drawn making equal angles with AB; EH and HF together are equal to CD.

Take O the centre, and join BO, and produce it; it will meet AE produced in G. Produce EH, FB to meet in I. Then since the angle $IHB=AHE=FHB$, and HB is perpendicular to FI, the triangles FHB, HBI are equal, and $FH=HI$. And since EG is parallel to FB, the angle $EGO=OBF$, and the vertical angles at O are equal, and $GO=OB$, $\therefore$ $EG=FB=BI$; whence $EI=GB$, and $\therefore$ EH, HF together are equal to EI *i. e.* to GB or CD the diameter of the circle.

(78.) *If from a point without a circle two straight lines be drawn, one of which touches and the other cuts the circle; a line drawn from the same point in any direction, equal to the tangent, will be parallel to the chord of the arc intercepted by two lines drawn from its other extremity to the former intersections of the circle.*

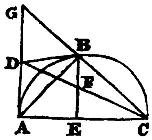


From the point A let AB, AD be drawn, of which AB touches the circle BCD, and AD cuts it, and draw $AE=AB$, in any direction; join CE, DE, cutting the circle in F and G; the chord FG will be parallel to AE.

Because (Eucl. iii. 36.) $DA : AB :: AB : AC$,
and $AE = AB$, $\therefore DA : AE :: AE : AC$,
i. e. the sides about the angle A of the triangles ADE, ACE are proportional, $\therefore$ (Eucl. vi. 6.) the triangles are equiangular, and the angle AEC is equal to the angle ADE. But since $CDGF$ is a quadrilateral figure in the circle, the angles CDG, CFG are equal to two right angles, *i. e.* to EFG, CFG, $\therefore CDG = EFG$, whence $AEF = EFG$, and FG is therefore parallel to AE.

* * *

(79.) *If from a point without a circle, two straight lines be drawn touching it, and from one point of contact a perpendicular be drawn to that diameter which passes through the other; this perpendicular will be bisected by the line joining the point without the circle and the other extremity of the diameter.*

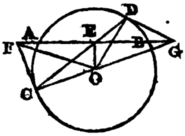


Let DA, DB be drawn from a point D without the circle ABC, touching it in A and B, and from B let BE be drawn perpendicular to AC the diameter passing through A; join CD; BE is bisected by CD in the point F.

For produce AD and CB to G; join AB. Then since $DA = DB$, the angle DAB is equal to the angle DBA. Now the angle ABG, being a right angle, is equal to BAG, BGA, of which $ABD = BAG$, $\therefore DBG = DGB$, and $DG = DB = DA$; and since AG is parallel to EB,
$BF : GD :: CF : CD :: EF : AD$,
and $GD = DA$, $\therefore BF = FE$.

* * *

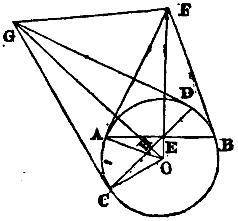
(80.) *If any chord in a circle be bisected by another, and produced to meet the tangents drawn from the extremities of the bisecting line; the parts intercepted between the tangents and the circumferences are equal.*



Let AB be bisected in E by CD; and to C and D let tangents be drawn, meeting AB produced in F and G; AF is equal to BG.

Find O the centre; join OC, OD, OE, OF, OG. Since OE is drawn from the centre to the point of bisection of AB (Eucl. iii. 3.) the angle OEF is a right angle; and the angle OCF is a right angle (Eucl. iii. 18.); $\therefore$ a circle may be described about $OEFC$. Also since ODG and OEG are right angles a circle may be described about $OEDG$; and the angle DOG is equal to the angle DEG in the same segment; but DEG is equal to FEC, *i.e.* to FOC, $\therefore DOG = FOC$; and ODG, OCF are equal being right angles, and $OC = OD$, $\therefore OF = OG$, and consequently $FE = EG$. But $AE = EB$, $\therefore FA = BG$.

(81.) *If one chord in a circle bisect another, and tangents drawn from the extremities of each be produced to meet; the line joining their points of intersection will be parallel to the bisected chord.*



Let AB be bisected by the line CD in E, and let the tangents AF, BF meet each other in F, and DG, CG in G. Join GF; GF is parallel to AB.

Join AO, CO, GO, FO; then

GO bisects CD in H, and OHE is a right angle; for the same reason FO passes through E, and AEO is a right angle. And since FAO is a right angle (Eucl. iii. 18.), and from A, AE is drawn perpendicular to the base,

(Eucl. vi. 8. Cor.) $FO : OA :: OA : OE$,

for the same reason,

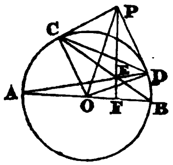
$(OC=)\,OA : OG :: OH : (OC=)\,OA$,

$\therefore$ *ex æquo per.* $FO : OG :: OH : OE$,

$\therefore$ the sides of the triangles FOG, OHE about the common angle O are proportional, and consequently the triangles are equiangular, and the angle GFO equal to EHO, and $\therefore$ a right angle, and equal to the alternate angle FEB, $\therefore$ AB is parallel to GF.

* * *

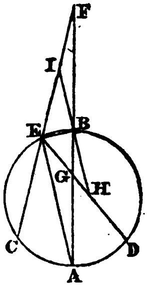
(82.) *If from a point without a circle two lines be drawn touching the circle, and from the extremities of any diameter lines be drawn to the points of contact, cutting each other within the circle; the line produced, which joins their intersection and the point without the circle, will be perpendicular to the diameter.*



From the point P without the circle ABC let there be drawn two tangents PC, PD. From A and B the extremities of a diameter, draw AD, BC to the points of contact, intersecting each other in E; join PE, and produce it to F; PF is perpendicular to AB.

Take O the centre; join CO, DO, CD, DB. Since CPD, COD, are together equal to two right angles, *CPD* is equal to *AOC, BOD* together, *i.e.* to twice *ADC, BCD* together, ∴ *CPD* is equal to the angle at the centre of a circle passing through *C, E,* and *D*; and since *PC*=*PD, P* is the centre itself; ∴ *PE*=*PD*, and the angle *PED* is equal to the angle *PDE*. But the angle *DBA*=*PDE*=*PED*=*AEF*, and the angle at *A* is common, ∴ *AFE*=*ADB*, and (Eucl. iii. 31.) is ∴ a right angle.

(83.) *If on opposite sides of the same extremity of the diameter of a circle equal arcs be taken, and from the extremities of these arcs lines be drawn to any point in the circumference, one of which cuts the diameter, and the other the diameter produced; the distances of the points of intersection from the extremities of the diameter are proportional to each other.*



On opposite sides of the point *A* in *AB* the diameter of the circle *ABC* let equal arcs *AC, AD* be taken; from *C* and *D* let *CE, DE* be drawn to any point *E* in the circumference, of which *CE* cuts *AB* produced in *F*, and *DE* cuts *AB* in *G*; then will *AF* : *FB* :: *AG* : *GB*.

Join *AE, BE*, and through *B* draw *HBI* parallel to *AE*. Since *AEB* is a right angle, *CEA* and *BEI* are together equal to a right angle, and ∴ equal to *AED, DEB*; and since *AC*=*AD*, *CEA* = *AED*, ∴ *BEI*=*BED*. Again since *AE* is parallel to *IH*, the angle *EIB* is equal to *CEA*=*AED*= the alternate angle *EHB*, ∴ the two triangles *EIB, EHB* having two angles in each equal, and one side *EB* common are equal, and $BI = BH$. And from the similar triangles AGE, BGH,

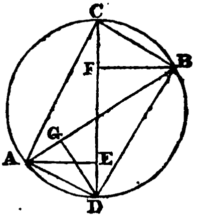
$$AG : AE :: BG : (BH =)\, BI,$$

$$\textit{alt.}\ AG : GB :: AE : BI :: AF : BF,$$

since the triangles AEF, FIB are similar.

* * *

(84.) *If from the extremities of any chord in a circle, perpendiculars be drawn to a diameter, and from either extremity of that diameter a perpendicular be drawn to the chord; it will divide it into segments, which are respectively mean proportionals between the segments of the diameter made by the perpendiculars.*



Let AB be any chord, and CD a diameter of the circle ABC; AE, BF perpendiculars from A and B to the diameter, and DG perpendicular from D to AB; the segment GB is a mean proportional between DF and CE; and AG a mean proportional between DE and CF.

Join AC, AD, BC, BD. Then the angle DBA being equal to DCA and the angles DGB, DAC, AEC being right angles, the triangles DGB, DAC, ACE are similar; also the triangles DBF, DBC are similar; whence

$$DF : DB :: DB : DC :: BG : AC$$

$$\text{but } DB : BG :: AC : CE,$$

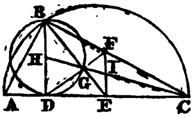
$$\therefore\ \textit{ex æquo}\ DF : BG :: BG : CE.$$

And in the same manner it may be proved that

$$DE : AG :: AG : CF.$$

* * *

(85.) *If from any point in the diameter of a semicircle, a perpendicular be drawn, meeting the circumference, and on it as a diameter a circle be described, to the centre of which a line is drawn from the farther extremity of the diameter of the semicircle, cutting its circumference; and through the point of intersection another line be drawn from the extremity of the perpendicular, meeting the diameter of the semicircle; this diameter will be divided into three segments which are in continued proportion.*



From any point D in the diameter AC of the semicircle ABC, let a perpendicular DB be drawn, on which describe a circle DBG. Find its centre H, join HC cutting the circumference in G; join BG and produce it to E; $AD : DE :: DE : EC$.

Join BC, and draw EF parallel to BD; join DG, GF, AB. Since EF is parallel to DB, and DB is bisected in H, $\therefore$ EF is bisected in I. Also the angle HBG is equal to the alternate angle GEI and $BHG = GIE$, $\therefore$ the triangles BHG, GIE are equiangular,

$$\text{and } BH : HG :: EI : IG,$$
$$\text{or } HD : HG :: FI : IG,$$

i. e. the sides about the equal angles are proportional, $\therefore$ the triangle FGI is equiangular to HDG, and the angle FGI is equal to HGD; whence HI being a straight line, FGD is also. Again the angle GDE is equal to the angle in the alternate segment DBG, whence the triangles BDE, FDE are similar,

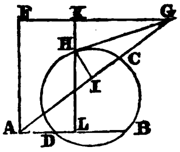
$$\therefore BD : DE :: DE : EF,$$

but $AD : DB :: EF : EC$, since the triangles *ADB*, *EFC* are similar,

$$\therefore AD : DE :: DE : EC.$$

* * *

(86.) *If from a point without a given circle, any two lines be drawn cutting the circle; to determine a point in the circumference, such that the sum of the perpendiculars from it upon these lines may be equal to a given line.*



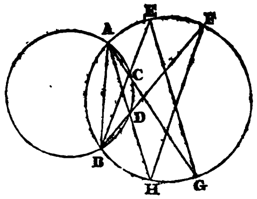
From the point *A* without the circle *BDC* let *AB*, *AC* be drawn cutting the circle; draw *AF* perpendicular to *AB*, and equal to the given line; *FG* parallel to *AB*, and meeting *AC* produced in *G*; from *G* draw *GH* bisecting the angle *AGF*, and (if the problem be possible) meeting the circle in *H*; *H* is the point required.

Through *H* draw *KL* perpendicular to *AB*, and *HI* perpendicular to *AC*; then the angle *KGH* being equal to *HGI*, and the angle at *K* to the angle at *I*, and the side *HG*, opposite to one of the equal angles in each common, *HK*=*HI*; whence *HI* and *HL* together are equal to *HK* and *HL* together, *i.e.* to *AF*, *i.e.* to the given line.

If *GH* cuts the circle, there are two points which answer the conditions.

* * *

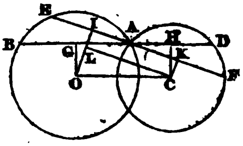
(87.) *If two circles cut each other, and any two points be taken in the circumference of one of them, through which lines are drawn from the points of intersection and produced to the circumference of the other; the straight lines joining the extremities of those which are drawn through the same point, are equal.*



Let the two circles *ACB*, *AEB* cut each other in *A* and *B*, and in *ACB* let any two points *C* and *D* be taken, through which draw *ACG*, *BCE*, *ADH*, *BDF*; and join *EG*, *FH*; *EG*=*FH*.

For the angles *CAD*, *CBD* being on the same circumference *CD* are equal to one another, ∴ the circumference *EF* is equal to the circumference *GH*. Add to each *FG*, and the circumference *EFG* is equal to *FGH*, ∴ (Eucl. iii. 29.), the straight line *EG*=*FH*.

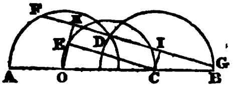
(88.) ***If two circles cut each other, the greatest line that can be drawn through the point of intersection is that which is parallel to the line joining their centres.***



Let the two circles *ABE*, *AFD* cut each other in *A*. Join *O*, *C* their centres, and through *A* let *BAD* be drawn parallel to *OC*; *BAD* is greater than any other line *EAF* which can be drawn through *A*.

Draw *OG*, *CH*, perpendicular to *BD*, and *OI*, *CK* perpendicular to *EF*. Then *AG* being half of *AB*, and *AH* of *AD*, *GH* is half of *BD*. For the same reason *IK* is half of *EF*. Draw *CL* parallel to *EF* and therefore at right angles to *OI*, and equal to *IK*. Then since the angle *CLO* is a right angle, it is greater than *COL*, ∴ the side *CO* is greater than *CL*, and *GH* than *IK*, consequently *BD* is greater than *EF*. In the same way *BD* may be shewn to be greater than any other line drawn through *A*.

(89.) *Having given the radii of two circles which cut each other, and the distance of their centres; to draw a straight line of given length through their point of intersection, so as to terminate in their circumferences.*

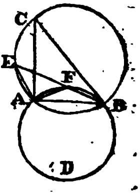


Let the two circles *AFD*, *BGD* cut each other in *D*; on *OC* the line joining their centres *O* and *C*, describe a semicircle *CEO*; and in it, from *C* place *CE* equal to half the given line, and through *D* draw *FDG* parallel to it; *FG* will be the line required.

Through *E* draw *OEH*, which (Eucl. iii. 31.) will be perpendicular to *FG*; and draw *CI* parallel to *OH*, and ∴ perpendicular to *DG*; then (Eucl. iii. 3.) *FD* and *DG* are bisected in *H* and *I*, and ∴ *FG* is double of *HI*; but *HECI* being a parallelogram, *HI* = *EC*; ∴ *FG* is double of *EC*, and consequently equal to the given line.

* * *

(90.) *If two circles cut each other; to draw from one of the points of intersection a straight line meeting the circles, so that the part of it intercepted between the circumferences may be equal to a given line.*

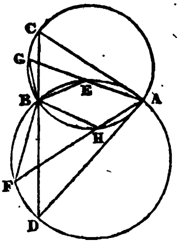


Let the two circles *ABC*, *ADB* cut each other in *A* and *B*. Join *AB*, and draw *BC* touching the circle *ABD*. Join *AC*; and take *AF* a fourth proportional to *BC*, *BA* and the given line; join *BF* and produce it to *E*; *BFE* will be the line required.

Since the angle *AFB* together with the angle in the segment *ADB* or (Eucl. iii. 32.) its equal *ABC*, are equal to two right angles, *i. e.* to the angles AFB, AFE, $\therefore ABC = AFE$ and $ACB = AEF$ being in the same segment, $\therefore$ the triangles ACB, AEF are equiangular,

and $AB : BC :: AF : FE$,
but $AB : BC :: AF :$ the given line,
whence FE is equal to the given line.

(91.) *If two circles cut each other; to draw from the point of intersection two lines, the parts of which intercepted between the circumferences may have a given ratio.*

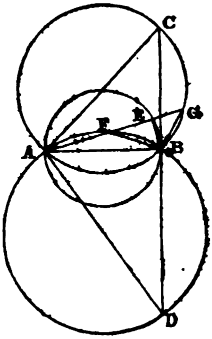


Let the two circles ABC, ABD cut each other in A and B; in the circle ABD place BE, BF which have to each other the given ratio; join AE, AF, and produce AE to G; EG will have to HF the given ratio.

Draw the diameters AC, AD; join GB, BH, BC, BD; then $ADBE$ being a quadrilateral figure inscribed in a circle, the angles AEB, ADB are equal to two right angles, and $\therefore$ equal to BEA, BEG, $\therefore BEG = BDA = BFA$. And since $AGBH$ is a quadrilateral figure inscribed in a circle, AHB, AGB are equal to two right angles, *i. e.* to AHB, BHF, $\therefore AGB = BHF$; hence the triangles GBE, FBH are equiangular,

$\therefore GE : HF :: BE : BF$, *i. e.* in the given ratio.

(92.) *If a semicircle be described on the common chord of two intersecting circles, and a line be drawn* *from one extremity of this chord cutting the two circles; the part intercepted between the two shall be divided by the semicircle into segments proportional to perpendiculars drawn in those circles from the other extremity of the chord.*



Let the two circles ACB, ADB cut each other in A and B; and on AB, the line joining the points of intersection, as a diameter describe the semicircle AEB, and draw any line $AFEG$ cutting the circumferences in F, E, G, and from B draw BC, BD perpendiculars to AB; then will $EF : EG :: BD : BC$.

Draw the diameters AC, AD; and join FB, EB, GB. Then $AFBD$ being a quadrilateral figure inscribed in a circle, the angles AFB, ADB are equal to two right angles, *i.e.* to AFB, BFE, $\therefore ADB = BFE$, and the angle FEB in a semicircle is equal to ABD, whence the triangles FEB, ABD are equiangular,

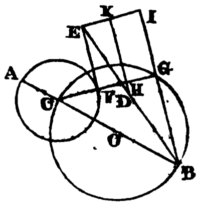
and $\therefore FE : EB :: DB : BA$.

Again, because the angle $AGB = ACB$, and BEG is a right angle, and $\therefore$ equal to ABC, the triangles EBG, ABC are equiangular, and

$$BE : EG :: AB : BC,$$
$$\text{but } FE : EB :: DB : BA,$$
$$\therefore \textit{ex æquo } FE : EG :: DB : BC.$$

* * *

(93.) *Two circles being given, the circumference of one of which passes through the centre of the other; to* *draw a chord from that centre such, that a perpendicular let fall upon it from a given point, may bisect that part of it which is intercepted between the circumferences.*



Let O and C be the centres of the two given circles, the circumference of the former passing through C, and let D be the given point. Join CO and produce it both ways to A and B. Join BD, and produce it to E, making $DE = DB$. Draw EF touching the circle AF in F; join CF and produce it to G; and on it let fall the perpendicular DH; then CG is the chord required, and FG is bisected in H.

Draw EI parallel to CG, meeting BG produced in I; produce DH to K. Then BG being perpendicular to CG (Eucl. iii. 31.), is parallel to DHK,

$$\therefore BD : DE :: IK : KE,$$

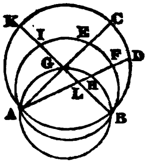
but $BD = DE$, $\therefore IK = KE$, whence $FH = HG$, and $\therefore FG$ is bisected in H.

Cor. If it be required to draw CG such, that the perpendicular DH may divide FG in any given ratio, take $DE : DB$ in that ratio, and proceed as in the proposition.

* * *

(94.) *If any number of circles cut each other in the same points, and from one of these points any number of lines be drawn; the parts of these which are intercepted between the several circumferences have the same ratio.*

Let any number of circles ABC, ABE, ABH cut each other in the same points A and B, and from A draw



AGEC, *AHFD*, meeting the circumferences; then *HF* : *GE* :: *FD* : *EC*.

Join *BG* and produce it to *K*;

then (Eucl. iii. 35.) *AL* : *LB* :: *LG* : *LH*,

and *AL* : *LB* :: *IL* : *LF*

and also :: *KL* : *LD*,

∴ (Eucl. v. 15.) *IL* : *LF* :: *GL* : *LH*,

and (Eucl. v. 19.) *IG* : *HF* :: *IL* : *LF*.

For the same reason, *IK* : *FD* :: *IL* : *LF*,

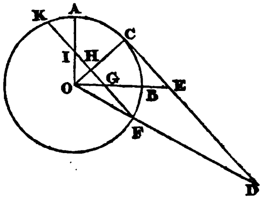
∴ *IG* : *HF* :: *IK* : *FD*.

In like manner, *GE* : *GI* :: *GB* : *GA* :: *GC* : *KG* :: *EC* : *IK*,

∴ *ex æquo* *GE* : *HF* :: *EC* : *FD*.

* * *

(95.) *In a given circle to place a straight line cutting two radii which are perpendicular to each other, in such a manner that the line itself may be trisected.*



Let *ABC* be the given circle, *AO* and *OB* being two radii at right angles to each other; bisect the angle *AOB* by *OC*; at *C* draw the tangent *CD*, and make it equal to 3 *CO*; produce *OB* to *E*; join *OD*, and from *F* draw *FGIK* parallel to *DC*; it will be trisected at the points *G* and *I*.

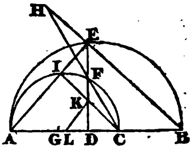
Since the angle at *C* is a right angle, and *COB* is half a right angle, ∴ also *CEO* is half a right angle, and equal to *COE*; whence *CO* = *CE*. And since *HF* is parallel to *CD*,

CE : *ED* :: *HG* : *GF*,

but *ED* is double of *EC*, ∴ *FG* is double of *HG*. But *HG* = *HI*, since *HO* bisects the angle *IOG*, and is perpendicular to IG; $\therefore FG = GI$. Also $HK = HF$, $\therefore IK = GF$;

whence $FG = GI = IK$, and FK is trisected.

(96.) *If a straight line be divided into any two parts, and upon the whole line and one of the parts, as diameters, semicircles be described; to determine a point in the less diameter, from which if a perpendicular be drawn cutting the circumferences, and the points of intersection and the extremities of the respective diameters be joined, and these lines produced to meet; the parts of them without the semicircles may have a given ratio.*



Let AB be divided into any two parts in the point C, and on AB, AC let semicircles be described. Take $AG : AC$ the duplicate of the given ratio, and make $CD : CB :: AG : GB$; D will be the point required.

From D draw the perpendicular DFE; join BE, CF, and produce them to H; join AE, CI, and from K draw KL parallel to EA.

Since $CD : CB :: AG : GB$,

comp. and *inv.* $DB : CD :: AB : AG$,

and since CI is parallel to BE, and KL to EA,

$BD : CD :: BE : CK :: BA : CL$,

whence (Eucl. v. 15.) $AB : AG :: AB : CL$,

$\therefore AG = CL$;

consequently $AG : AC :: CL : CA$,

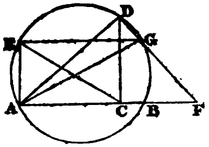
i.e. in the duplicate ratio of $CK : CF$,

or (by similar triangles) of $HE : HF$.

But $AG : AC$ is the duplicate of the given ratio,
∴ $GHE : HF$ is in the given ratio.

* * *

(97.) *If a straight line be divided into any two parts, and from the point of section a perpendicular be erected, which is a mean proportional between one of the parts and the whole line, and a circle described through the extremities of the line and the perpendicular; the whole line, the perpendicular, the aforesaid part, and a perpendicular drawn from its extremity to the circumference will be in continued proportion.*



Let AB be divided into any two parts in C, and from C draw the perpendicular CD equal to a mean proportional between AB and AC; and through A, B, D let a circle be described, and draw AE perpendicular to AB; AB, CD, AC, AE are in continued proportion.

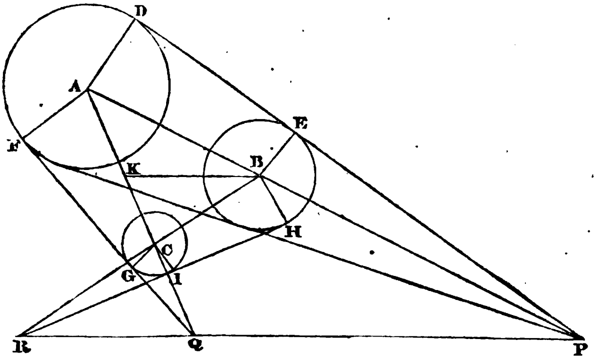
In AB produced take $BF = AC$. Join FD meeting the circumference in G; join AG, AD, GE. Then because $BF = AC$, ∴ $CF = AB$, and CD is a mean proportional between AC and CF, ∴ ADG is a right angle, whence (Eucl. iii. 21.) AEG is also a right angle, and equal to EAC; ∴ EG is parallel and equal to AB, *i. e.* to CF; whence (Eucl. i. 33.) EC and GF are equal and parallel, and the angle $ACE = CFD = ADC$, and the triangles AEC, ADC, CDF are similar,

$$\therefore (CF=)\,AB : CD :: CD : CA :: CA : AE.$$

* * *

(98.) *If the tangents drawn to every two of three unequal circles be produced till they meet; the points of intersection will be in a straight line.*

Let *A, B, C* be the centres of the three circles; and let *DE, FG, HI* be respectively tangents to each of two



circles, meeting the lines joining the centres in the points *P, Q, R*; *P, Q, R* are the points in which two tangents to the circles would intersect. Join *PQ, QR*; they are in the same straight line.

Join *AD, AF, BE, BH, CG, CI,* and draw *BK* parallel to *PQ*. Then *BE* and *AD* being perpendicular to *PD* are parallel,

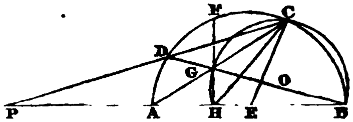
∴ *AD* : *BE* :: *AP* : *BP*,
or *AF* : *BH* :: *AP* : *BP* :: *AQ* : *QK*.
But (*CG* =) *CI* : *AF* :: *CQ* : *AQ*,
∴ *ex æquo CI* : *BH* :: *CQ* : *QK*.
But *CI* : *BH* :: *CR* : *BR*,
∴ (Eucl. v. 15.) *CR* : *BR* :: *CQ* : *QK*,
and *CR* : *CB* :: *CQ* : *CK*;

also the vertically opposite angles at *C* are equal, ∴ the triangles *CBK, CQR* are similar, and the angle *CQR* (Eucl. vi. 6.) is equal to *BKC*, ∴ *CQR* and *CQP* are together equal to BKC, CQP, *i. e.* (Eucl. i. 29.) to two right angles, whence (Eucl. i. 14.) PQ and QR are in the same straight line.

* * *

(99.) *If from the extremities of the diameter of a circle any number of chords be drawn, two and two intersecting each other in a perpendicular to that diameter; the lines joining the extremities of every corresponding two will meet the diameter produced in the same point.*

From A and B, the extremities of the diameter AB of a semicircle, let AC, BD be drawn intersecting each other in FH, which is perpendicular to AB. Join CD, and produce it to meet BA in P; P is a fixed point, or the line joining the extremities of every other two chords intersecting each other in FH will pass through P.



Join BC; and bisect BG in O; and with the centre O, and radius OB, describe a circle HGB, which will circumscribe the quadrilateral figure $HGCB$. Take E the centre of the semicircle, and join HC, EC. The angle PCE is equal to PCA, ACE together, *i. e.* to DBA, CAE together; and the angle CHE is equal to ACH, CAH together, *i.e.* to DBA, CAH together, $\therefore PCE = CHE$, and the angle at E being common, the triangles CEH, CPE are equiangular;

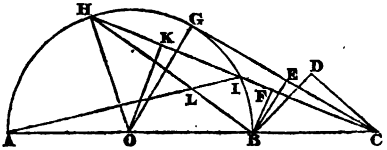
whence $EH : EC :: EC : EP$,

in which proportion the three first terms being invariable, EP is also, and the point E being fixed, P is also.

* * *

(100.) *If from a given point in the diameter of a semicircle produced, three straight lines be drawn, one of which is inclined at a given angle to the diameter, another touches the semicircle, and the third cuts it, in such a manner, that the distance of the given point from the nearer extremity of the diameter, and the perpendiculars drawn from that extremity on the three aforesaid lines may be proportional; then will the lines, which join the extremities of the diameter and of that part of the cutting line which is within the circle, intersect each other in an angle equal to the given angle.*

From a given point *C*, in the diameter *AB* produced of the semicircle *AGB*, draw *CD* inclined at a given angle to *AC*, *CG* touching, and *CIH* cutting the circle in such a manner that *BD*, *BE*, *BF* being drawn respectively perpendicular to them, *CB* may be to *BD* as *BE* to *BF*; then if *AI*, *BH* be joined, the angle *ALH* or *BLI* will be equal to *BCD*.



Join *OH*, *OG*; and draw *OK* perpendicular to *HI*. Now the angles at *E* and *F* being right angles, as also those at *G* and *K*, *BE* is parallel to *OG*, and *BF* to *OK*;

$$\therefore (OG=)\ OH : BE :: CO : CB :: OK : BF,$$

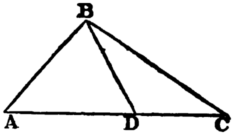
$$\therefore OH : OK :: BE : BF :: BC : BD;$$

also the angle at *D* is equal to *OKH*, ∴ (Eucl. vi. 7.) the triangles *OHK*, *BCD* are equiangular, and the angle *OHK* is equal to *BCD*. But *OHB* is equal to *OBH*, *i. e.* to *AIH* (Eucl. iii. 21.), ∴ *OHK* is equal to *AIH*, *LHI* together, *i. e.* to *ALH* (Eucl. i. 32.); wherefore *ALH* is equal to *BCD*.

The Straight line, and triangles

SECT. III.

(1.) *Any side of a triangle is greater than the difference between the other two sides.*



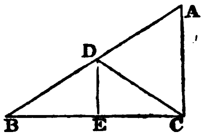
Let *ABC* be a triangle; any of its sides is greater than the difference of the other two.

Let *AC* be greater than *AB*; and cut off $AD=AB$; join *BD*; then the angle *ABD* is equal to *ADB*. But the exterior angle *BDC* is greater than *DBA*, *i. e.* than *BDA*, and ∴ greater than *DBC* (Eucl. i. 16.); whence *BC* is greater than *DC*, *i. e.* than the difference of the sides *AC* and *AB*. In the same way it may be shewn that *AB* is greater than the difference of *AC* and *BC*; and *AC* greater than the difference of *AB* and *BC*.

* * *

(2.) *In any right-angled triangle, the straight line joining the right angle and the bisection of the hypothenuse is equal to half the hypothenuse.*

Let *ACB* be a right-angled triangle, whose hypo-

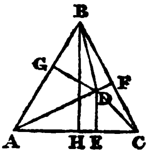


thenuse AB is bisected in D; join DC; DC is equal to the half of AB.

From D draw DE parallel to AC, $\therefore$ (Eucl. vi. 2.) $BE=EC$, and ED is common and at right angles to BC, $\therefore$ $DC=BD$, *i.e.* the half of AB.

* * *

(3.) *If from any point within an equilateral triangle perpendiculars be drawn to the sides; they are together equal to a perpendicular drawn from any of the angles to the opposite side.*



From any point D within the equilateral triangle ABC let perpendiculars DE, DF, DG be drawn to the sides, they are together equal to BH a perpendicular drawn from B on the opposite side AC.

Join DA, DB, DC. Since triangles upon the same and equal bases are to one another as their altitudes,

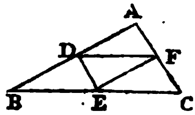
$$ABC : ADC :: BH : DE,$$
$$\text{also } ABC : BDC :: BH : DF,$$
$$\text{and } ABC : ADB :: BH : DG;$$

whence $ABC : ADC+BDC+ADB :: BH : DE+DF+DG$, in which proportion the first term being equal to the second, $\therefore$ $DE+DF+DG=BH$.

* * *

(4.) *If the points of bisection of the sides of a given triangle be joined; the triangle so formed will be one fourth of the given triangle.*

Let the sides of the triangle *ABC* be bisected in the points *D*, *E*, *F*; join *DE*, *EF*, *FD*; the triangle *DEF* is one fourth of the triangle *ABC*.

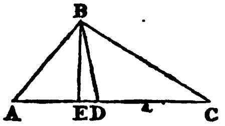


Since *AB* and *AC* are bisected in *D* and *F*, (Eucl, vi. 2.) *DF* is parallel to *BC*; and for the same reason *FE* is parallel to *AB*, and *DFEB* is a parallelogram, ∴ the triangle *DFE* is equal to *DBE*. In the same way it may be shewn to be equal to *FEC* and *ADF*; and ∴ it is one fourth of *ABC*.

* * *

(5.) *The difference of the angles at the base of any triangle is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.*

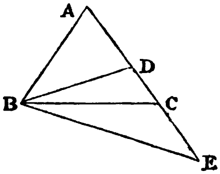
From *B* the vertex of the triangle *ABC* let *BE* be drawn perpendicular to the base, and *BD* bisecting the angle *ABC*; the difference of the angles *BAC*, *BCA* is double the angle *EBD*.



The angle *BAC* is equal (Eucl. i. 32.) to the difference of the angles *BEC* and *ABE*, *i. e.* of a right angle and *ABE*. Also the angle *BCA* is equal to the difference of a right angle and *EBC*, ∴ the difference of the angles *BAC* and *BCA* is equal to the difference of the angles *ABE* and *EBC*, *i. e.* (since *ABD*=*DBC*) to twice the angle *EBD*.

* * *

(6.) *If from one of the equal angles of an isosceles triangle any line be drawn to the opposite side, and from the same point, a line be drawn to the opposite side produced, so that the part intercepted between them may be equal to the former; the angle contained by the side of the triangle and the first drawn line is double of the angle contained by the base and the latter.*

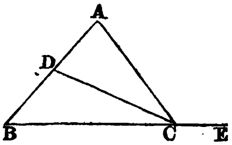


Let *ABC* be an isosceles triangle, having the side *AB* equal to *AC*. From *B* draw any line *BD*, and also *BE* cutting off *DE* equal to *DB*; the angle *ABD* is double of *CBE*.

For the angle *DCB* is equal to the two *DEB*, *CBE*, *i. e.* to the two *DBE*, *CBE*, or to *DBC* and twice *CBE*; but *DCB* is equal to *ABC*, ∴ *ABC* is equal to *DBC* and twice *CBE*, and taking away the angle *DBC*, which is common to both, the angle *ABD* is equal to twice *CBE*.

* * *

(7.) *If from the extremity of the base of an isosceles triangle, a line equal to one of the sides be drawn to meet the opposite side; the angle formed by this line and the base produced, is equal to three times either of the equal angles of the triangle.*

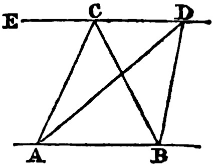


Let *ABC* be an isosceles triangle having the side *AB* equal to *AC*. From *C* to *AB* (produced if necessary) draw *CD* equal to *AC*, and let *BC* be produced; the angle *DCE* is equal to three times the angle *ABC*.

Since *CA* is equal to *CD*, the angle *CAD* is equal to *CDA*, ∴ *CDA* and twice *ABC* are together equal to two right angles, and ∴ are equal to *CDA*, *CDB*; whence *CDB* is double of *ABC*. Now (Eucl. i. 32.) the angle *DCE* is equal to the two angles *CDB*, *CBD* and consequently is equal to three times the angle *ABC*.

* * *

(8.) *The sum of the sides of an isosceles triangle is less than the sum of the sides of any other triangle on the same base and between the same parallels.*

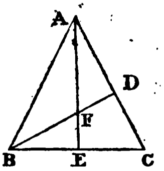


Let *ACB* be an isosceles triangle, and *ADB* any other triangle on the same base, and between the same parallels *AB*, *ED*; *AC* and *CB* together will be less than *AD* and *DB*.

Since *EC* is parallel to *AB*, the angle *ECA* is equal to *CAB*; and for the same reason *DCB* is equal to *CBA*; but *CAB* being equal to *CBA*, *ECA* is equal to *DCB*; ∴ *AC* and *BC* drawn from two given points *A* and *B* on the same side of the line *ECD* given in position make equal angles with the line, ∴ (i. 6.) they are together less than any other two lines *AD*, *DB*, drawn from the same points to that line.

* * *

(9.) *If from one of the equal angles of an isosceles triangle a perpendicular be drawn to the opposite side; the part of it intercepted by a perpendicular from the vertex will have to one of the equal sides, the same ratio that the segment of the base has to the perpendicular upon the base.*



Let ABC be an isosceles triangle, having the side AB equal to AC. From B and A let fall perpendiculars BD, AE; then will $BF : AC :: BE : EA$.

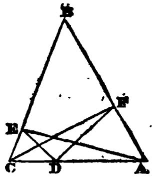
Since the angles BDA, AEC are right angles, and the angle DAF common to the two triangles FAD, EAC, $\therefore$ the triangles are similar. But the triangle BFE is similar to AFD, and $\therefore$ to EAC;

whence $BF : BE :: AC : AE$,

and $BF : AC :: BE : EA$.

* * *

(10.) *If from any point in the base of an isosceles triangle lines be drawn to the opposite sides, making equal angles with the base; the triangles formed by these lines, the segments of the base, and the lines joining the intersections of the sides and the angles opposite, will be equal.*

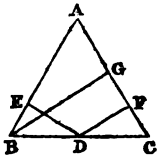


From any point D in AC the base of the isosceles triangle ABC let DE, DF be drawn making the angles CDE, ADF equal to one another; join AE, CF; the triangles AED, CDF are equal.

Since the angle ADF is equal to the angle EDC, and $FAD = ECD$, the triangles ECD, FAD are equiangular, and $AD : DC :: FD : DE$. Also since the angle FDA is equal to EDC, add to each the angle FDE, $\therefore$ the angle $ADE = CDF$; hence the sides about the equal angles are reciprocally proportional, and $\therefore$ (Eucl. vi. 15.) the triangles ADE, FCD are equal.

* * *

(11.) *If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides; these together shall be equal to a perpendicular drawn from either extremity of the base to the opposite side.*



Let ABC be an isosceles triangle, from any point D in the base of which, let DE, DF be drawn perpendicular to the sides; and from B let BG be drawn perpendicular to AC; BG is equal to DE and DF together.

Since the angle EBD is equal to the angle at C, and the angles at E and F are right angles, the triangles BED, DFC are equiangular, and

$$\therefore BD : DC :: DE : DF,$$

$$comp.\ BC : DE + DF :: DC : DF.$$

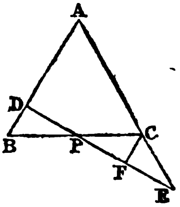
But BG being parallel to DF, $DC : DF :: BC : BG$,

$$\text{whence } BC : BG :: BC : DE + DF,$$

$$\text{and } \therefore BG = DE + DF.$$

* * *

(12.) *Of all triangles having the same vertical angle, and whose bases pass through a given point, the least is that whose base is bisected in the given point.*

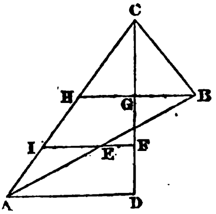


Let BAC be the vertical angle of any number of triangles, whose bases pass through a given point P, and let BC be bisected in P: ABC is less than any other triangle ADE.

From C draw CF parallel to AB; then the angle DBP is equal to PCF, and the vertically opposite angles DPB, CPF are equal, and $BP = PC$,

∴ the triangle DBP is equal to the triangle PCF, and ∴ DPB is less than CPE; add to each the trapezium $ADPC$, and ABC is less than ADE. In the same manner ABC may be proved to be less than any other triangle whose base passes through P.

(13.) *If from the angles at the base of a triangle perpendiculars be let fall on a line which bisects the vertical angle; the part of this line intercepted between these perpendiculars will be bisected by a perpendicular from the middle of the base.*



From A and B let perpendiculars AD, BG be drawn to the line CD which bisects the angle at C; the part GD will be bisected by a perpendicular EF from E the middle point of the base AB.

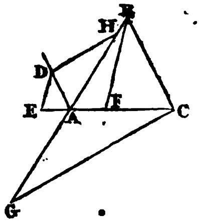
Produce BG, FE to H and I. Then IE being parallel to HB, and $AE = EB$, ∴ (Eucl. vi. 2.) $AI = IH$. Also since AD is parallel to HG and IF,

$$DF : FG :: AI : IH,$$

whence $DF = FG$, and DG is bisected in F.

(14.) *If from one of the angles at the base of a triangle a line be drawn parallel to the opposite side, and from any point in it lines be drawn making any angles with the sides (produced, if necessary); they will have the same ratio that lines have, which are drawn parallel to them from the other angles, and terminated by the same sides.*

From A one of the angles of the triangle ABC let AD be drawn parallel to BC the opposite side; and from any point D in it, let DE, DH be drawn making any angles with the sides; draw BF, CG parallel to them respectively; $DE : DH :: BF : CG$.



Since DE is parallel to BF, and DA to BC, the triangles DEA, BFC are equiangular,

$$\therefore DE : DA :: BF : BC;$$

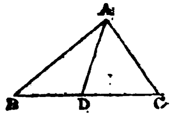
and in a similar manner it may be shewn, that

$$DA : DH :: BC : CG,$$

$$\therefore DE : DH :: BF : CG.$$

(15.) *To bisect a given triangle by a line drawn from one of its angles.*

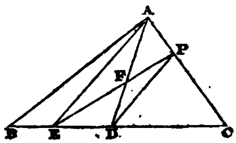
Let ABC be the given triangle, and A the angle, from which the bisecting line is to be drawn. Bisect the opposite side AC in D, and join AD; AD bisects the triangle.



For the bases BD, DC being equal, (Eucl. i. 38.) the triangles ABD, ADC are also equal.

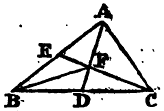
(16.) *To bisect a given triangle by a line drawn from a given point in one of its sides.*

Let ABC be the given triangle, and P the given point. Bisect BC in D, join AD, PD; and from A draw AE parallel to PD; join PE; PE bisects the triangle ABC.



Since AE is parallel to PD, the triangle APD is equal to the triangle EPD; from each of them take away the triangle PFD, and $AFP = EFD$. Also since BD is equal to DC, the triangle ABD is equal to the triangle ADC; parts of which EFD, AFP are equal, ∴ $ABEF$ is equal to $PFDC$; whence $ABEF$ and AFP together, or $ABEP$ will be equal to $PFDC$ and FED together, *i. e.* to PEC; and ∴ the triangle ABC is bisected by PE.

(17.) *To determine a point within a given triangle, from which lines drawn to the several angles, will divide the triangle into three equal parts.*

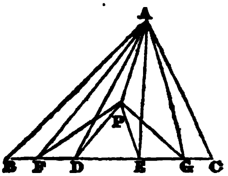


Let ABC be the given triangle; bisect AB, BC, in E, and D; join AD, CE, BF; F is the point required.

Since $BD = DC$, the triangle BAD is equal to DAC; and for the same reason the triangle BFD is equal to DFC; ∴ the triangle BFA is equal to AFC. Again, since $BE = EA$, the triangle BEC is equal to the triangle AEC; parts of which, the triangles BEF, AEF are equal; ∴ the triangle BFC is equal to AFC; and ∴ the three BFC, BFA, AFC are equal to one another.

(18.) *To trisect a given triangle from a given point within it.*

Let ABC be the given triangle, and P the given point within it. Trisect the side BC in D and E; join

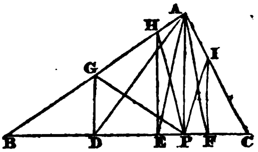


PD, *PE*; and from *A* draw *AF*, *AG* respectively parallel to them. Join *PF*, *PG*, *AP*. Those three lines will divide the triangle into three equal parts.

Join *AD*, *AE*. Since *AF* is parallel to *PD*, the triangle *APF* is equal to *ADF*; to each of these add *ABF*, ∴ *APFB* is equal to *ADB*. In the same manner *APGC* is equal to *AEC*; and ∴ the remainder *FPG* is equal to *DAE*. Now the triangles *ABD*, *ADE*, *AEC*, being on equal bases and of the same altitude, are equal, ∴ *APFB*, *PFG*, *APGC* are also equal; and the triangle *ABC* is trisected.

* * *

(19.) *From a given point in the side of a triangle, to draw lines, which will divide the triangle into parts which shall have a given ratio.*

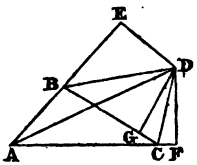


Let *ABC* be the given triangle, and *P* the given point in the side *BC*. Divide *BC*, in the points *D*, *E*, *F*, into parts which shall have the given ratio. Join *AD*, *AE*, *AF*, *AP*; and draw *DG*, *EH*, *FI* parallel to *AP*. Join *PG*, *PH*, *PI*; they will divide the triangle, as required.

For the triangles *ABD*, *ADE*, *AEF*, *AFC* being as their bases will be in the given ratio. And since *DG* is parallel to *AP*, the triangles *DGA*, *DGP* are equal, ∴ *DBA*, *GPB* are equal. And since the triangle *ADP* $=$ *AGP*, and *AEP* $=$ *AHP*, ∴ *ADE* $=$ *HPG*. Also *APE* $=$ *AHP*, and *APF* $=$ *AIP*, ∴ *AEF* $=$ *AHPI*, and

$AFC=PIC$; $\therefore$ the parts PBG, GPH, $HPIA$, IPC are equal to ABD, ADE, AEF, AFC, and are $\therefore$ in the given ratio. The same may be proved whatever be the number of parts.

(20.) *If two exterior angles of a triangle be bisected, and from the point of intersection of the bisecting lines, a line be drawn to the opposite angle of the triangle; it will bisect that angle.*



Let the exterior angles EBC, BCF, of the triangle ABC, be bisected by the lines BD, CD meeting in D. Join DA; it will bisect the angle BAC.

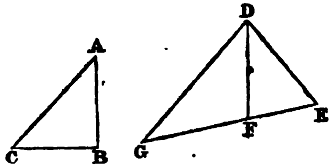
Let fall the perpendiculars DE, DF, DG. Then the angles DBE, DBG being equal, and the angles at E and G being right angles, and DB common to the triangles DBE, DBG, $\therefore DE=DG$. In the same manner $DG=DF$; and $\therefore DE=DF$. Hence in the right-angled triangles DAE, DAF, DE is equal to DF and DA is common, $\therefore$ the triangles are equiangular, and the angles DAE DAF are equal, *i. e.* BAC is bisected by AD.

(21.) *If in two triangles the vertical angle of the one be equal to that of the other, and one other angle of the former be equal to the exterior angle at the base of the latter; the sides about the third angle of the former shall be proportional to those about the interior angle at the base of the latter.*

Let *ABC*, *DEF* be two triangles having the angle *BAC* equal to *EDF*, and *ABC* equal to the exterior angle *DFG*, made by producing the side *EF*;

then *AC* : *CB* :: *DE* : *EF*.

At the point *D* in the line *FD*, make the angle *FDG* equal to the angle *EDF* or *BAC*, and meeting *EF* produced in *G*. Since the angle *FDG* is equal to the angle *BAC*, and *DFG* is equal to *ABC*, ∴ the triangles *ABC*, *DFG* are equiangular, and



AC : *CB* :: *DG* : *GF*.

But since the angle *GDE* is bisected by *DF*, ∴ (Eucl. vi. 3.)

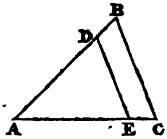
DG : *GF* :: *DE* : *EF*,

∴ *AC* : *CB* :: *DE* : *EF*.

* * *

(22.) *In a given triangle to draw a line parallel to one of the sides, so that it may be a mean proportional between the segments of the base.*

Let *ABC* be the given triangle; in the base of which take a point *E*, such that *AE* may be to *EC* in the duplicate ratio of *AC* : *CB*; draw *ED* parallel to *BC*; *ED* is the line required.



Since *ED* is parallel to *BC*, *AE* : *ED* :: *AC* : *CB*. But *AE* : *EC* in the duplicate ratio of *AC* : *CB*, and therefore also in the duplicate ratio of $AE : ED$; whence from the definition of the duplicate ratio,

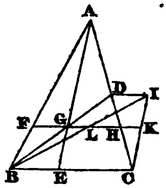
$$AE : ED :: ED : EC.$$

* * *

(23.) *To draw a line parallel to the common base of two triangles which have different altitudes, so that the parts of it intercepted by the sides may have a given ratio.*

Let ABC, DBC be two triangles on the same base BC, the vertex D being in the side AC. Divide BC in E, so that $BC : CE$ may be equal to the given ratio. Join AE, cutting BD in G; and through G draw FH parallel to BC; FH is the line required.

Since FH is parallel to BC, $FH : GH :: BC : CE$, *i. e.* in the given ratio.

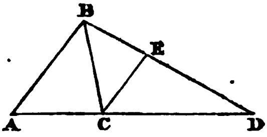


But if the vertex I is not in AC, draw ID parallel to BC; join BD; divide the base BC, as before; join AE, and draw FK parallel to BC. Then it is evident that $GH = LK$, and $\therefore FH : LK$ in the given ratio.

Cor. If the triangles be upon equal bases, but in the same straight line, the line may be drawn in a similar manner.

* * *

(24.) *If the base of a triangle be produced, so that the whole may be to the part produced in the duplicate ratio of the sides; the line joining the vertex and the extremity of the part produced will be a mean proportional between the whole line produced and the part produced.*

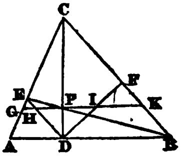


Let *AC* be produced to *D*, so that *AD* may be to *DC* in the duplicate ratio of *AB* : *BC*; join *BD*; it will be a mean proportional between *AD* and *DC*.

Draw *CE* parallel to *AB*; then *AB* : *CE* :: *AD* : *DC*, *i.e.* in the duplicate ratio of *AB* : *BC*, whence *AB* : *BC* :: *BC* : *CE*, *i.e.* the sides about the equal angles *ABC*, *BCE* are proportional; therefore the triangles *ABC*, *BCE* are similar, and the angle at *A* is equal to the angle *CBD*; ∴ the triangles *ABD*, *CBD* are equiangular, and

AB : *BD* :: *BD* : *DC*.

(25.) *To determine a point within a given triangle, which will divide a line parallel to the base into two segments, such that the excess of each segment above the perpendicular distance between the parallel lines may be to each other in the duplicate ratio of the respective segments.*



Let *ABC* be the given triangle. From *C* draw *CD* perpendicular to *AB*, and from *D* draw *DE*, *DF* bisecting the angles *ADC*, *BDC*. Join *BE*, cutting *CD* in *P*; *P* is the point required.

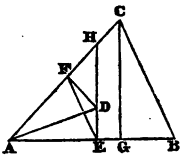
Through *P* draw *GHIK* parallel to *AB*; then the angle *PDH* is equal to the angle *HDA*, *i.e.* to the alternate angle *PHD*; and ∴ *HP*, and in like manner *PI* will each be equal to *PD* the perpendicular distance of *GK* from *AB*; and *GH*, *IK* will be equal to the excess of each segment above that distance PD. And since GP is parallel to AB,

$$GP : PK :: AD : DB :: GH : (HP =) PI,$$

hence (Eucl. v. 19. Cor.) $GH : (PI =)\ PH :: PH : IK$, and $\therefore$ $GH : IK$ in the duplicate ratio of $GH : HP$, *i. e.* of $GP : PK$.

* * *

(26.) *If perpendiculars be drawn to two sides of a triangle from any two points therein; the distance of their concourse from that of the two sides will be to the distance between the two points, as either side is to the perpendicular drawn from its extremity upon the other.*



From any two points E, F in the sides AB, AC of the triangle ABC, let perpendiculars ED, FD be drawn, meeting in D. Join AD, EF, and from C draw CG perpendicular to AB; $AD : FE :: AC : CG$.

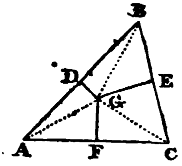
Produce ED to H. And since the angles AED, AFD are right angles, a circle described on AD as a diameter will pass through F and E, and $\therefore$ the angles FAD, FED standing in the same segment are equal; $\therefore$ the triangles AHD, HEF are equiangular;

and $\therefore$ $AD : FE :: AH : HE :: AC : CG$,

since HE is parallel to CG.

* * *

(27.) *If the three sides of a triangle be bisected, the perpendiculars drawn to the sides at the three points of bisection, will meet in the same point.*



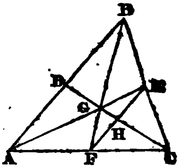
Let the sides of the triangle ABC be bisected in the points D, E, F. Draw the perpendiculars EG, FG meeting in G. The perpendicular at D also passes through G.

Join GD, GA, GB, GC. Since $AF = FC$, and FG is common to the triangles AFC, CFG, and the angles at F are right angles, $\therefore AG = GC$. In the same way it may be shewn that $GC = GB$; $\therefore AG = GB$; but $AD = DB$, and DG is common to the triangles ADG, BDG, $\therefore$ the angles at D are equal and $\therefore$ right angles, or the perpendicular at D passes through G.

Cor. The point of intersection of the perpendiculars is equally distant from the three angles.

* * *

(28.) *If from the three angles of a triangle lines be drawn to the points of bisection of the opposite sides, these lines intersect each other in the same point.*



Let the sides of the triangle ABC be bisected in D, E, F. Join AE, CD, meeting each other in G. Join BG, GF; BGF is a straight line.

Join EF, meeting CD in H. Then (Eucl. vi. 2.) FE is parallel to AB, and $\therefore$ the triangles DAG, GEH are equiangular,

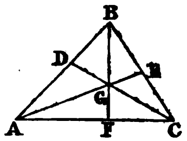
$$\therefore DA : DG :: HE : HG,$$

$$\text{or } DB : DG :: HF : HG,$$

i.e. the sides about the equal angles are proportional; $\therefore$ the triangles BDG, GHF are similar, and the angle $DGB = HGF$; and $\therefore BG$ and GF are in the same straight line.

* * *

(29.) *The three straight lines, which bisect the three angles of a triangle, meet in the same point.*

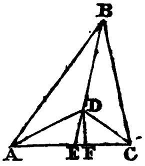


Let the angles BAC, BCA be bisected by the lines AE, CD, and through G their point of intersection draw BGF; it bisects the angle at A.

For (Eucl. vi. 3.) $BC : CF :: BG : GF :: BA : AF$,
$\therefore BC : BA :: CF : FA$,
or FB bisects the angle ABC.

* * *

(30.) *If the three angles of a triangle be bisected, and one of the bisecting lines be produced to the opposite side; the angle contained by this line produced, and one of the others is equal to the angle contained by the third, and a perpendicular drawn from the common point of intersection of the three lines to the aforesaid side.*

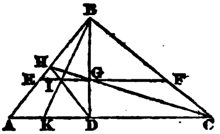


Let the three angles of the triangle ABC be bisected by the lines AD, BD, CD; produce BD to E, and from D draw DF perpendicular to AC; the angle ADE is equal to CDF.

Since the three angles of the triangle ABC are equal to two right angles, $\therefore$ the angles DAB, DBA, DCF are together equal to one right angle, *i. e.* to DCF, and CDF; whence the two angles DAB, DBA are together equal to the angle CDF; but ADE is equal to the same two angles, and $\therefore ADE$ is equal to CDF.

* * *

(31.) *In a right-angled triangle, if a straight line be drawn parallel to the hypothenuse, and cutting the* *perpendicular drawn from the right angle; and through the point of intersection a line be drawn from one of the acute angles to the opposite side, and the extremity of this line and of the perpendicular be joined; the locus of its intersection with the line parallel to the hypothenuse will be a straight line.*



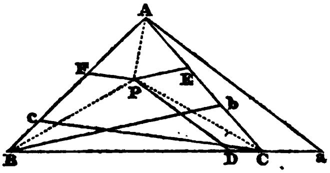
Let EF be drawn parallel to AC the hypothenuse of the right-angled triangle ABC; and from the right angle B let the perpendicular DB be drawn, meeting EF in G; through G draw CGH; join HD; the locus of I, the intersection of EF and HD is a straight line.

Because EG is parallel to AC the base of the triangles AHC, ABD, $AK : KD :: EI : IG :: AD : DC$. But AD and DC are invariable, $\therefore$ the ratios of $AK : KD$, and $EI : IG$ are also. In the same manner if any other line be drawn parallel to the hypothenuse, and a similar construction be made, the point of intersection will divide the part intercepted between AB and BD in the ratio of $AD : DC$, or $AK : KD$, and will $\therefore$ be in the line BK, which is the locus required.

(32.) *If from the angles of a triangle, lines, each equal to a given line, be drawn to the opposite sides (produced if necessary); and from any point within, lines be drawn parallel to these, and meeting the sides of the triangle; these lines shall together be equal to the given line.*

From the angles of the triangle ABC let the lines Aa, Bb, Cc be drawn to the opposites sides, each equal to a given line L; and parallel to them respectively draw, from any point P, the lines PD, PE, PF; these together will be equal to L.

Join PA, PB, PC. Then since the triangles ABC, APC are on the same base AC, they are to one another



as the perpendiculars from B and P, *i.e.* by similar triangles, as $Bb : PE$, or as $L : PE$. In the same way,

$$ABC : ABP :: L : PF,$$

$$\text{and } ABC : BPC :: L : PD;$$

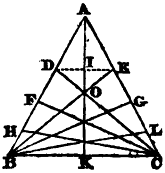
$$\therefore ABC : APC+ABP+BPC :: L : PE+PF+PD;$$

and since the first term is equal to the second, the third will be equal to the fourth, or $L=PD+PE+PF$.

* * *

(33.) *If the sides of a triangle be cut proportionally, and lines be drawn from the points of section to the opposite angles; the intersections of these lines will be in the same line, viz. that drawn from the vertex to the middle of the base.*

Let the sides of the triangle ABC be cut proportionally, so that $AD : AE :: DF : EG :: FH : GL :: HB : LC$. Join BE, BG, BL, CD, CF, CH; these lines will intersect each other in the line AK drawn from A to K the middle of the base BC.

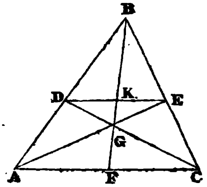


Join *DE*. Then since when any number of magnitudes are proportional, as one antecedent is to its consequent, so are all the antecedents taken together to all the consequents together, $\therefore AD : AE :: AB : AC$, and *DE* is parallel to *BC*. Join *KO*, and let it meet *DE* in *I*. The triangles *BOK*, *IOE* are similar, and therefore,

$BK : KO :: EI : IO$, and for the same reason,

$CK : KO :: DI : IO$, whence $EI = DI$, and *DE* is bisected by *KO*; and it is also bisected by *AK*, $\therefore$ *AK* passes through *O*. In the same manner it may be shewn that *BG* and *CF*, as also *BI*, *CH* intersect each other in points which are in the line *AK*.

(34.) *If from any point in one side of a triangle, two lines be drawn, one to the opposite angle, and the other parallel to the base, and the former intersect a line drawn from the vertex bisecting the base; this point of intersection, that of the line parallel to the base and the third side, and the third angular point are in the same straight line.*



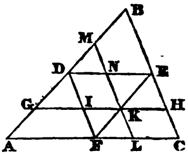
From any point *D* in the side *AB* of the triangle *ABC*, let *DE* be drawn parallel to *AC*, and *DC* joined; and let *DC* meet *BF* drawn from *B* to the middle of *AC* in *G*; *A*, *G*, *E* are in the same straight line.

Let *DE* cut *BF* in *K*. The triangles *DGK*, *CGF* are equiangular, and

$$\therefore DG : GC :: DK : FC :: DE : AC;$$

hence the triangles DGE, AGC, having one angle in each equal, viz. EDG, GCA, and the sides about them proportional, are therefore similar; whence the angles AGC, DGE are equal; and DGC being a straight line, AGE is also.

(35.) *If one side of a triangle be divided into any two parts, and from the point of section two straight lines be drawn parallel to, and terminating at the other sides, and the points of termination be joined; and any other line be drawn parallel to either of the two former lines, so as to intersect the other, and to terminate in the sides of the triangle; then the two extreme parts of the three segments into which the line so drawn is divided will always be in the ratio of the segments of the first divided line.*



Let AB be divided into any two parts in D, from which draw DE, DF parallel to the other two sides of the triangle; join EF, and draw GH parallel to DE, meeting DF and EF in I and K; $GI : KH :: AD : DB$; and if LM be parallel to DF,

$$LK : MN :: AD : DB.$$

Since GI is parallel to AF, and NK to DF,

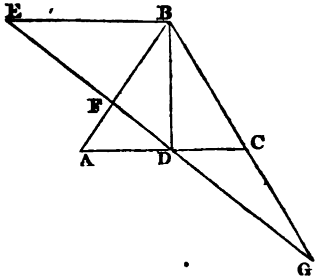
$$GI : AF :: (ID =)\ NK : FD :: NE : DE :: KH : FC,$$

$\therefore GI : KH :: AF : FC :: AD : DB$, since DF is parallel to BC. Again since ML is parallel to BC,

$$MN : BE :: ND : DE :: KF : FE :: KL : EC,$$

$$\therefore MN : KL :: BE : EC :: BD : DA.$$

(36.) *If through the point of bisection of the base of a triangle any line be drawn, intersecting one side of the triangle, and the other produced, and meeting a parallel to the base from the vertex; this line will be cut harmonically.*



From the vertex B of the triangle ABC, let BE be drawn parallel to the base AC, and through the middle point D let any line EGF be drawn meeting AB, BC, BE, in F, G, E;

$$EG : DG :: FE : FD.$$

Since AD is parallel to BE,

$$FE : FD :: BE : AD,$$

$$\text{but } BE : (DC=)\ DA :: EG : GD,$$

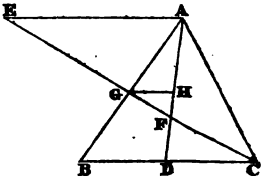
since the triangles BGE, DGC are equiangular,

$$\therefore \text{(Eucl. v. 15.)}\ EG : GD :: FE : FD,$$

or the line is divided harmonically.

* * *

(37.) *If from either angle of a triangle a line be drawn intersecting that which joins the vertex and the bisection of the base, the opposite side, and the line from the vertex parallel to the base; it will be cut harmonically.*



From the vertex A of the triangle ABC, let AE be drawn parallel to the base BC, and AD to its point of bisection D; and from C draw any line $CFGE$; then will

$$CE : CF :: EG : FG.$$

Draw GH parallel to BC. Since AE and BC are parallel, (Eucl. vi. 2.)

$$BA : AG :: CE : EG,$$

and since GH is parallel to BD,

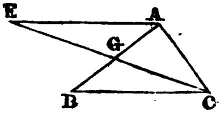
$$BA : AG :: BD : GH :: DC : GH,$$
$$:: CF : FG,$$

since the triangles DFC, GHF are similar;

$$\therefore CE : EG :: CF : FG,$$

$$\text{and } CE : CF :: EG : FG.$$

(38.) *To draw a line from one of the angles at the base of a triangle, so that the part of it cut off by a line drawn from the vertex parallel to the base, may have a given ratio to the part cut off by the opposite side.*



From A let AE be drawn parallel to BC. Divide AB in G, so that $AB : AG$ in the given ratio; join CG, and produce it to meet AE in E. CGE is the line required.

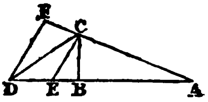
For the triangles AGE, BGC are equiangular,

$$\therefore CG : EG :: BG : AG,$$

whence (Eucl. v. 18.) $CE : EG :: BA : AG$, *i. e.* in the given ratio.

(39.) *To determine that point in the base produced of a right-angled triangle, from which the line drawn to the angle opposite to the base shall have the same ratio to the base produced, which the perpendicular has to the base itself.*

Let AB be the base, and CB the perpendicular of



a right-angled triangle. Draw *CE* at right angles to *AC*, meeting *AB* produced in *E*. At the point *C* make the angle *ECD* = *CAB*. *D* is the point required.

From *D* draw *DF* perpendicular to *AC*, and ∴ parallel to *CE*. Since the angle *FDC* is equal to the alternate angle *DCE*, *i. e.* to *CAB*, and the angles at *F* and *B* are right angles, ∴ the triangles *DCF*, *ACB* are equiangular; and *DAF* is also equiangular to *ACB*, hence

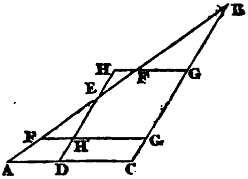
$$FD : DA :: BC : CA,$$

$$\text{and } DC : DF :: AC : AB$$

$$\therefore \textit{ex æquo per. } CD : DA :: CB : BA.$$

* * *

(40.) *If the base of any triangle be divided into two parts by a line which is a mean proportional between them, and which being drawn parallel to the second side is terminated in the third; any line parallel to the base will be divided by the mean proportional (produced if necessary) into segments, which will be to each other inversely as the whole mean proportional to that segment which is terminated in the third side of the triangle.*



Let *AC* the base of the triangle *ABC* be divided into two parts in *D*, by a line *DE* which is parallel to *BC*, and a mean proportional between *AD* and *DC*; then any line *FG* parallel to *AC*, and meeting *DE* (produced if necessary) in *H*, will be divided into segments *FH*, *HG*, which are to each other inversely as the lines *DE*, *HE*.

For since FH is parallel to AD,

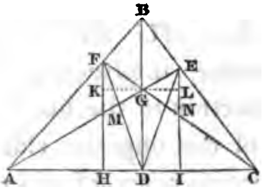
$FH : AD :: HE : DE$,

but $AD : DE :: DE : (DC =) HG$,

$\therefore FH : DE :: HE : HG$.

(41.) *If from the extremities of the base of any triangle, two straight lines be drawn intersecting each other in the perpendicular, and terminating in the opposite sides; straight lines drawn from thence to the intersection of the perpendicular with the base, will make equal angles with the base.*

From A and C, the extremities of AC, the base of the triangle ABC, let AE, CF be drawn intersecting the perpendicular BD in the same point G. Join FD, ED; these lines make equal angles FDA, EDC with the base.



Draw EI, FH perpendicular, and KGL parallel to the base, then FH is parallel to BD,

and $\therefore BG : BD :: FM : FH$.

And in the same manner it may be shewn that

$BG : BD :: EN : EI$;

whence $FM : FH :: EN : EI$;

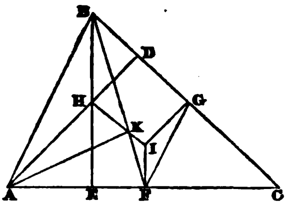
and $\therefore FM : EN :: FH : EI$.

But $FM : EN :: FG : GN :: KG : GL :: HD : DI$,

$\therefore HD : DI :: HF : EI$,

whence the two triangles DFH, DEI, having the angle at H equal to the angle at I, and the sides about the equal angles proportional, are equiangular; $\therefore$ the angle HDF is equal to EDI.

(42.) ***In every triangle, the intersection of the perpendiculars drawn from the angles to the opposite sides, the intersection of the lines from the angles to the middle of the opposite sides, and the intersection of the perpendiculars from the middle of the sides, are all in the same straight line. And the distances of those points from one another are in a given ratio.***



From the angles *A* and *B* of the triangle *ABC*, let *AD*, *BE* be drawn perpendicular to the opposite sides, *H* will be the intersection of the three perpendiculars (vii. 34.). From *A* and *B* draw *AG*, *BF* to the points of bisection of the opposite sides, intersecting in *K*, which ∴ is (iii. 28.) the intersection of the lines drawn from the angles to the middle of the opposite sides; and from *F* and *G* draw the perpendiculars *FI*, *GI* meeting in *I*, which ∴ (iii. 27.) is the intersection of the three perpendiculars. Join *HK*, *KI*; *HKI* is a straight line.

Join *GF*. (Eucl. vi. 2.) *AB* is parallel to, and double of *GF*; ∴ by similar triangles *ABK*, *KFG*, *BK* is double of *KF*, and *AK* double of *KG*. And the triangles *AHB*, *FIG* are equiangular, ∴ *AH* is double of *IG*, and *BH* is double of *IF*;

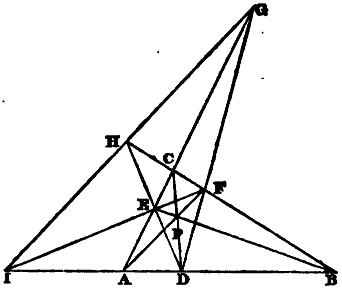
and ∴ *BH* : *IF* :: 2 : 1 :: *BK* : *KF*,

whence the triangles *BHK*, *KIF* having the angles at *B* and *F* equal, and the sides about them proportional, are similar, ∴ the angle *HKB* is equal to *IKF*, ∴ *H*, *K*, and *I* are in the same straight line.

And since *BK* is double of *KF*, *HK* is double of *KI*, and ∴ their distances from each other will be in an invariable ratio.

(43.) *If straight lines be drawn from the angles of a triangle through any point, either within or without the triangle, to meet the sides, and the lines joining these points of intersection and the sides of the triangle be produced to meet; the three points of concourse will be in the same straight line.*

Let ABC be a triangle from the three angles of which let lines AF, BE, CD be drawn through a point P within the triangle. Join DE, DF, EF, and produce them to meet the sides in H, G, I; these three points will be in the same straight line.

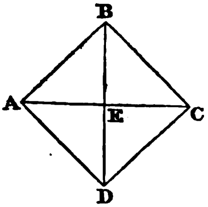


Join GH, HI. Then the three angles of the triangle DHG being equal to two right angles, as also the three EHI, EIH, and (HEI or) DEF, as also the two DFI, GFI; ∴ the three angles of the triangle DHG together with the angles EHI, EIH, DEF, DFI, GFI are equal to six right angles. Now the angles of the triangles DEF, FGI are together equal to four right angles, whence DHG, DHI are equal to two right angles; or GH, HI are in the same straight line.

Squares, parallelograms, and polygons.

SECT. IV.

(1.) *The diameters of a rhombus bisect each other at right angles.*

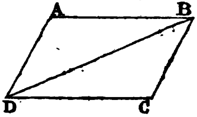


Let $ABCD$ be a rhombus, whose diameters are AC, BD; they bisect each other at right angles in E.

Since $AB = AD$, and AC is common to the two triangles ABC, ADC, the two BA, AC are equal to the two DA, AC, each to each, and $BC = DC$, $\therefore$ the angle BAC is equal to the angle DAC. Again, since BA, AE are equal to DA, AE, each to each, and the included angles are equal, $\therefore$ $BE = ED$, and the angles AEB, AED are equal, and $\therefore$ are right angles. For the same reason $AE = EC$; also the angles BEC, DEC are right angles.

* * *

(2.) *If the opposite sides or opposite angles of a quadrilateral figure be equal, the figure will be a parallelogram.*

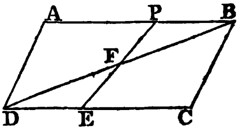


Let $ABCD$ be a quadrilateral figure, whose opposite sides are equal. Join BD. Since $AB = DC$, and BD is common, the two AB, BD are equal to the two CD, DB, each to each, and $AD = BC$, $\therefore$ the angle $ABD = BDC$, whence (Eucl. i. 27.) AB is parallel to DC; also the angle $ADB = DBC$, whence AD is parallel to BC; and the figure is a parallelogram.

Again, let the opposite angles be equal. Then since the four angles of the quadrilateral figure *ABCD* are equal to four right angles, and that *BAD*, *ADC* together are equal to *DCB*, *CBA*, ∴ *BAD*, *ADC* together are equal to two right angles; whence *AB* is parallel to *CD*. In the same way it may be shewn that *AD* is parallel to *BC*, and ∴ *ABCD* is a parallelogram.

* * *

(3.) *To bisect a parallelogram by a line drawn from a point in one of its sides.*



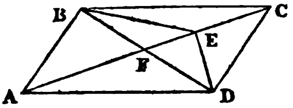
Let *ABCD* be a parallelogram, and *P* a given point in the side *AB*. Draw the diameter *BD*, which bisects the parallelogram. Bisect *BD* in *F*; join *PF*, and produce it to *E*. *PE* bisects the parallelogram.

Since the angle *PBD* is equal to the angle *BDE*, and the vertically opposite angles at *F* are equal, and *BF* = *FD*, ∴ the triangles *PBF*, *DFE* are equal. But the triangle *ABD* is equal to *BDC*, ∴ *APFD* is equal to *BFEC*; and to these equals adding the equal triangles *DFE*, *PFB*, the figure *APED* = *PECB*; and *AC* is ∴ bisected by *PE*.

Cor. Any line drawn through the middle point of the diameter of a parallelogram is bisected in that point.

* * *

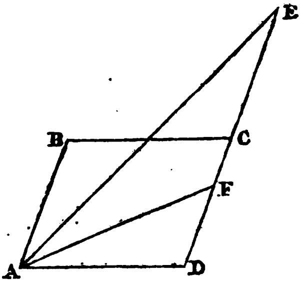
(4.) *If from any point in the diameter (or diameter produced) of a parallelogram straight lines be drawn to the opposite angles; they will cut off equal triangles.*



From any point E, in AC the diameter of the parallelogram $ABCD$, let lines EB, ED be drawn; the triangles ABE, AED are equal; as also the triangles BEC, CED.

Draw the diameter BD. The bases BF, FD being equal, the triangles BFA, DFA (Eucl. i. 38.), as also the triangles BFE, DFE are equal, hence $\therefore$ BAE, DAE are equal. And ABC being equal to ADC, the triangles BEC, DEC are also equal.

(5.) *From one of the angles of a parallelogram to draw a line to the opposite side, which shall be equal to that side together with the segment of it which is intercepted between the line and the opposite angle.*



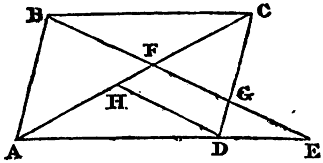
Let $ABCD$ be the parallelogram, A the angle from which the line is to be drawn. Produce DC to E, making $CE = CD$. Join AE, and at the point A make the angle $EAF = AEF$; AF is the line required.

For CE being equal to CD, $EF = DC$ and CF together; and the angles FEA, FAE being equal, $FA = FE$, and $\therefore$ $AF = DC$ and CF together.

Cor. In the same manner if $CE = CB$, $AF = EF = BC$ and CF together.

(6.) *If from one of the angles of a parallelogram a straight line be drawn cutting the diameter and a side*

produced; the segment intercepted between the angle and the diameter, is a mean proportional between the segments intercepted between the diameter and the sides.



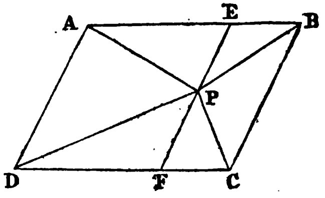
From B, one of the angles of the parallelogram $ABCD$, let any line BE be drawn cutting the diameter AG in F, the opposite side in G, and AD produced in E; BF is a mean proportional between FG and FE.

Draw DH parallel to BF, and ∴ equal to it; ∴ also $AH = FC$, and $AF = CH$. Since HD is parallel to BG, $FG : DH$ (:: $CF : CH$ :: $AH : AF$) :: $DH : FE$,

or $FG : BF :: BF : FE$.

* * *

(7.) *The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of two opposite sides, are together half of the parallelogram.*

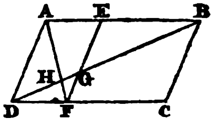


Let P be any point within the parallelogram $ABCD$, from which let lines PA, PD, PB, PC be drawn to the extremities of the opposite sides; the triangles PAD, PBC are equal to half the parallelogram; as also the triangles APB, DPC.

Through E draw EPF parallel to AD or BC; then (Eucl. i. 41.) the triangle APD is half of $AEFD$, and BPC is half of $BEFC$, ∴ APD, BPC are together half of *ABCD*. In the same manner if a line be drawn through *P* parallel to *AB* or *DC*, it may be shewn that *APB*, *DPC* together are half of *ABCD*.

* * *

(8.) *If a straight line be drawn parallel to one of the sides of a parallelogram, and one extremity of this line be joined to the opposite one of the parallel side, by a line which also cuts the diameter; the segments of the diameter made by this line will be reciprocally proportional to the segments of that part of it which is intercepted between the side and the parallel line.*



Let *EF* be drawn parallel to *AD* one of the sides of the parallelogram *ABCD*, cutting the diameter *BD* in *G*. Join *AF*, cutting it also in *H*; then will *BH* : *HD* :: *HD* : *HG*.

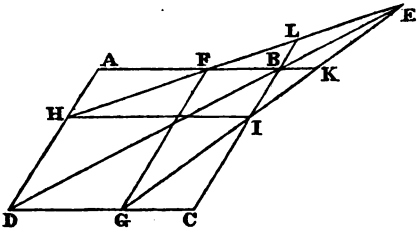
For the angle *ABH* being equal to *HDF*, and *AHB* *DHF*, the triangles *AHB*, *DHF* are equiangular, and ∴ *BH* : *HD* :: *AH* : *HF* :: *DH* : *HG*, since the triangles *AHD*, *FHG* are also equiangular.

* * *

(9.) *If two lines be drawn parallel and equal to the adjacent sides of a parallelogram; the lines joining their extremities, if produced, will meet the diameter in the same point.*

Let *HI*, *FG* be drawn equal and parallel to the adjacent sides *AB*, *BC* of the parallelogram *ABCD*. Join *HF*, *GI*; these lines produced will meet the diameter *DB* in the same point.

Produce *AB*, *CB* to *K* and *L*. Then the triangles *AFH*, *LBF* having the vertically opposite angles at *F* equal, and the alternate angles *AHF*, *FLB* also equal, are equiangular,



whence $AF : FB :: (HA =) IB : BL$,

and in the same manner it may be shewn that

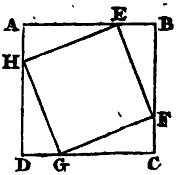
$(GC =) FB : CI :: BK : BI$,

$\therefore AF : CI :: BK : BL$.

But $AF = DG$, and $CI = DH$, $\therefore DG : DH :: BK : BL$, and $\therefore$ *HF*, *DB*, *GI* converge to the same point.

* * *

(10.) *If in the sides of a square, at equal distances from the four angles, four other points be taken, one in each side; the figure contained by the straight lines which join them shall also be a square.*

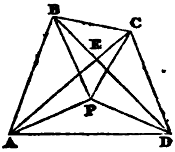


Let *E*, *F*, *G*, *H* be four points at equal distances from the angles of the square *ABCD*. Join *EF*, *FG*, *GH*, *HE*; *EFGH* is also a square.

Since $AH = EB$, and $AE = BF$, and the angles at *A* and *B* are right angles, $\therefore HE = EF$, and the angle *AEH* is equal to the angle *BFE*. In the same way it may be shewn that *HG* and *GF* are each of them equal to HE and EF, $\therefore$ the figure $HEFG$ is equilateral. It is also rectangular; for since the exterior angle FEA is equal to the interior angles EBF, EFB; parts of which AEH and EFB are equal; $\therefore$ the remaining angle FEH is equal to the remaining angle FBE, and $\therefore$ is a right angle. In the same manner it may be shewn that the angles at F, G, H are right angles, and $\therefore$ $EFGH$ being equilateral and rectangular, is a square.

* * *

(11.) *The sum of the diagonals of a trapezium is less than the sum of any four lines which can be drawn to the four angles from any point within the figure, except from the intersection of the diagonals.*



Let $ABCD$ be a trapezium, whose diagonals are AC, BD, cutting each other in E; they are less than the sum of any four lines which can be drawn to the angles from any other point within the trapezium.

Take any point P, and join PA, PB, PC, PD. Then (Eucl. i. 20.) AC is less than AP, PC; and BD is less than BP, PD; $\therefore$ AC, BD are less than AP, PB, PC, PD.

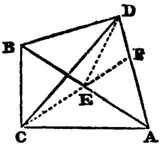
* * *

(12.) *Every trapezium is divided by its diagonals into four triangles proportional to each other.*

Let $ABCD$ be a trapezium (see last Fig.) divided by its diagonals AC, BD into the triangles AEB, BEC, AED, DEC; these are proportional to each other.

$$\begin{aligned} &\text{For (Eucl. vi. 1.)} \quad AEB : BEC :: AE : EC, \\ &\text{and} \quad AED : DEC :: AE : EC, \\ &\therefore AEB : BEC :: AED : DEC. \end{aligned}$$

(13.) *If two opposite angles of a trapezium be right angles; the angles subtended by either side at the two opposite angular points will be equal.*



Let the two angles *ACB, ADB* of the trapezium *ACBD*, be right angles. Join *AB, CD*; the angles *ACD, ABD*, subtended by *AD*, are equal.

Bisect *AB* in *E*. Join *CE, ED*, and produce *CE* to *F*. Then (iii. 2.) *AE, EB, EC, ED* are equal to one another. Also the angle *AED* is equal to the two *EDB, EBD*, *i.e.* to twice *EBD*; and *DEF* is equal to the two *DCE, EDC*, *i.e.* to twice *DCE*; and *AEF* is equal to twice *ACE*; ∴ twice *ACE* and twice *ECD*, or twice *ACD* will be equal to *AED*, *i. e.* to twice *EBD*, $\therefore ACD = ABD$.

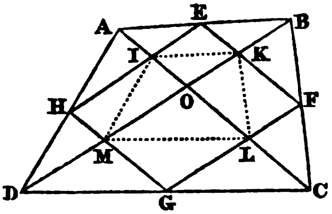
The same may be proved for the angles standing on any of the other sides.

(14.) *To determine the figure formed by joining the points of bisection of the sides of a trapezium; and its ratio to the trapezium.*

Let *ABCD* be a trapezium, whose sides are bisected in *E, F, G, H*. Let the points of bisection be joined; and draw the diagonals *AC, BD*.

Since *AB, AD* are bisected in *E* and *H*, (Eucl. vi. 2.)

EH is parallel to *BD*; and for the same reason *FG* is parallel to *BD*, and ∴ to *EH*. In the same way it may be shewn that *EF* is parallel to *HG*, and ∴ the figure *EFGH* is a parallelogram.



Again (Eucl. vi. 19.) the triangle *EBF* is to the triangle *ABC* in the duplicate ratio of *EB* : *AB*, *i. e.* in the ratio of 1 : 4, ∴ *EBF* is equal to one fourth of *ABC*; for the same reason *HDG* is one fourth of *DAC*, whence *EBF* and *HDG* are together equal to one fourth of the trapezium. For the same reason *HAE* and *GFC* are together equal to one fourth of the trapezium; therefore the four triangles together are equal to half the trapezium; and consequently *HEFG* is equal to half of *ABCD*.

Cor. 1. Hence two lines, drawn to bisect the opposite sides of a trapezium, will also bisect each other.

Cor. 2. If the sides of a square be bisected and the points of bisection joined, the inscribed figure is a square, and equal to half the original square.

* * *

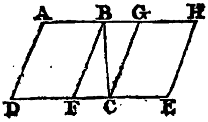
(15.) *To determine the figure formed by joining the points where the diagonals of the trapezium cut the parallelogram; and its ratio to the trapezium.*

Let *I*, *K*, *L*, *M* be the points of intersection; (see last Fig.) join *IK*, *KL*, *LM*, *MI*. And let *O* be the intersection of the diagonals. Since *EK* is parallel to *OI*,

BK : *KO* :: *BE* : *EA*, i. e. in a ratio of equality. For the same reason *AI*=*IO*. Whence the sides of the triangle *AOB* being cut proportionally, *IK* is parallel to *AB*. In the same manner it may be shewn that *KL*, *LM*, *MI* are respectively parallel to *BC*, *CD*, *DA*; whence the figure *IKLM* will be similar to *ABCD*. Also since the triangle *MIK* is half the parallelogram *HK*, and *MLK* half of *GK*, ∴ the figure *IKLM* is half of *HF*, and ∴ equal to one fourth of the trapezium *ABCD*.

* * *

(16.) *If two sides of a trapezium be parallel; its area is equal to half that of a parallelogram, whose base is the sum of those two sides, and altitude the perpendicular distance between them.*

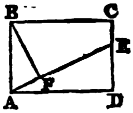


Let *ABCD* be a trapezium, whose side *AB* is parallel to *DC*. Produce *DC* to *E*, making *CE*=*AB*. Draw *BF*, and *CG*, *EH* parallel to *AD*, meeting *AB* produced. Then *AE* is a parallelogram of the same altitude with the trapezium, and its base is equal to the sum of the sides *AB*, *DC*; and *ABCD* is half of *AE*.

Since *DF*=*CE*, the parallelograms *AF*, *GE* are equal (Eucl. i. 36.); and the diameter *BC* bisects the parallelogram *FG*; whence *ABCD*=*BCEH*, and ∴ the trapezium is half of the parallelogram *AE*.

* * *

(17.) *If from any angle of a rectangular parallelogram a line be drawn to the opposite side, and from the adjacent angle of the trapezium thus formed another be drawn perpendicular to the former; the rectangle contained by these two lines, is equal to the given parallelogram.*



From *A*, one of the angles of the rectangular parallelogram *ABCD*, let any line *AE* be drawn to the opposite side, cutting off the trapezium *ABCE*; from *B* let fall the perpendicular *BF*; the rectangle contained by *AE, BF*, is equal to the parallelogram *ABCD*.

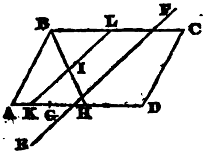
For *BA* being parallel to *ED*, the angle *BAF* = *AED*, and the angles at *F* and *D* are right angles, ∴ the triangles *BAF*, *AED* are similar; whence

BF : *BA* :: *AD* : *AE*,

and the rectangle *AE, BF* is equal to the rectangle *BA, AD*, *i.e.* to *ABCD*.

* * *

(18.) *To divide a parallelogram into two parts which shall have a given ratio, by a line drawn parallel to a given line.*



Let *ABCD* be the given parallelogram, and *EF* the line whose direction is given. Divide *AD* in *G* in the given ratio, and make *GH* = *AG*. Join *BH*; bisect it in *I*, and through *I* draw *KL* parallel to *EF*; *KL* divides the parallelogram in the given ratio.

For (Eucl. vi. 1. and i. 41.)

ABH : *ABCD* :: *AG* : *AD*.

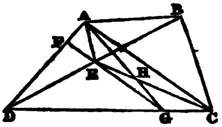
But the triangles *BIL*, *KIH* are equal, since *BI*=*IH*, and *BL* is parallel to *KH*; ∴ *ABH*=*ABLK*, and

ABLK : *ABCD* :: *AG* : *AD*,

∴ *ABLK* : *LKDC* :: *AG* : *GD*, *i. e.* in the given ratio.

* * *

(19.) *To bisect a trapezium by a line drawn from one of its angles.*



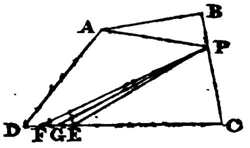
Let *ABCD* be the given trapezium, and *A* the angle from which it is to be bisected. Draw the diagonals *AC*, *BD*; and bisect *BD*, which is opposite to the angle *A*, in the point *E*. Join *AE*, *CE*; and through *E* draw *FEG* parallel to *AC*. Join *AG*; *AG* bisects the trapezium.

Since *DE* is equal to *EB*, the triangles *AED*, *AEB* are equal; as also *DEC*, *BEC*; ∴ the figure *AECD* is equal to the figure *AECB*. Also (Eucl. i. 38.) the triangles *AEG*, *CEG* are equal; take away ∴ the common part *EHG*, and *AEH*=*GHC*. To the figure *AECD* add *AEH*, and take away its equal *GHC*; and to *AECB* add *GHC*, and take away *AEH*; and the triangle *AGD* is equal to the trapezium *AGCB*, or the given trapezium is bisected by *AG*.

* * *

(20.) *To bisect a trapezium by a line drawn from a given point in one of its sides.*

Let *ABCD* be the given trapezium, and *P* the given point.

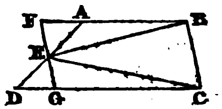


Join *PA*, and from the angle *P* bisect the trapezium *APCD* (iv. 19.) by the line *PE*. On *PE* make the triangle *PEF* equal to *ABP*. Bisect *EF* in *G*; join *PG*. *PG* bisects the trapezium.

Since *FG* is equal to *GE*, the triangle *PGF* is equal to the triangle *PGE*. But *PGE* is equal to half the triangle *ABP*; and *PEC* is half the figure *PABC*; whence *PGC* is half of the trapezium *ABCD*; which is ∴ bisected by *PG*.

* * *

(21.) *If two sides of a trapezium be parallel; the triangle contained by either of the other sides, and the two straight lines drawn from its extremities to the bisection of the opposite side, is half the trapezium.*

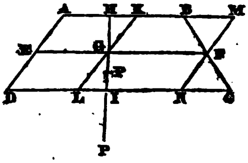


Let *ABCD* be a trapezium, having the side *AB* parallel to *DC*. Let *AD* be bisected in *E*; join *BE*, *CE*; the triangle *BEC* is half of the trapezium.

Through *E* draw *FEG* parallel to *BC*, meeting *CD* in *G*, and *BA* produced in *F*. The alternate angles *FAE*, *EDG* being equal, as also the angles at *E*, and *AE*=*ED*, ∴ the triangles *AEF*, *DEG* are equal; whence the parallelogram *BFGC* is equal to the trapezium *ABCD*. But *BFGC* and the triangle *BEC*, being on the same base *BC*, and between the same parallels *BC*, *FG*, the triangle *BEC* is half of *BFGC*, and ∴ also half of *ABCD*.

Cor. From the demonstration it appears, that a trapezium which has two sides parallel, may be reduced to a parallelogram equal to it, by drawing through the point of bisection of one of the sides, which are not parallel, a line parallel to the other of those sides, and meeting the parallel sides.

(22.) *To divide a given trapezium whose opposite sides are parallel, in a given ratio, by a line drawn through a given point, and terminated by the two parallel sides.*



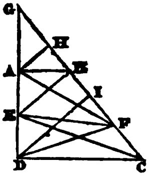
Let $ABCD$ be a trapezium whose sides AB, DC are parallel, and P the given point. Bisect AD in E, and draw EF parallel to AB, meeting BC in F. Divide EF in G in the given ratio; and through G and P draw IPH; IPH will divide the trapezium in the given ratio.

Draw KGL, MFN parallel to AD; then $EA = KG = MF$, and $ED = GL = FN$; but $AE = ED$, $\therefore KG = GL$, and $MF = FN$; whence (iv. 21. Cor.) $ADIH = AL$, and $HICB = KN$.

$$\text{Now } AL : KN :: EG : GF,$$
$$\therefore ADIH : HICB :: EG : GF,$$

i. e. in the given ratio.

(23.) *If a trapezium, which has two of its adjacent angles right angles, be bisected by a line drawn from the middle of one of those sides which are not parallel; the* *sum of the parallel sides will have to one of them the same ratio, that the side which is not bisected has to that segment of it which is adjacent to the other.*



Let $ABCD$ be a trapezium, having the angles at A and D right angles, and $\therefore$ AB, DC parallel; and let the trapezium be bisected by EF; if AD be bisected in E,

$$AB+DC : AB :: BC : CF;$$

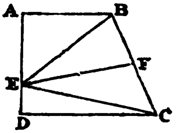
but if BC be bisected in F,

$$AB+DC : AB :: AD : DE.$$

Produce DA, CB to meet in G. Join AF, DF, BE, CE, and let fall the perpendiculars AH, DI. Since $AE=ED$, the triangles AFE, DFE are equal, $\therefore$ the triangles DFC, AFB are equal; $\therefore$ the rectangles FC, DI and BF, AH are equal,

$$\text{and } FC : FB :: AH : DI :: AB : CD,$$

$$\therefore AB+CD : AB :: (FC+FB=)\ BC : FC.$$



But if BC be bisected, the triangles EBF, ECF being equal, the triangles AEB, EDC are also equal,

$\therefore$ (Eucl. vi. 15.) $AB : CD :: DE : EA$,

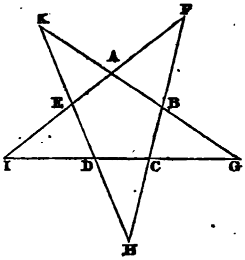
and $AB+CD : AB :: (DE+EA=)\ AD : DE$.

In like manner $AB+DC : DC :: AD : AE$.

* * *

(24.) *If the sides of an equilateral and equiangular pentagon be produced to meet; the angles formed by these lines are together equal to two right angles.*

Let $ABCDE$ be an equilateral and equiangular

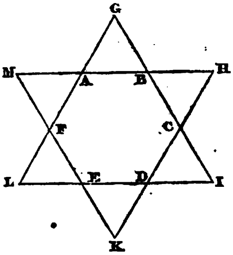


pentagon; and let the sides be produced to meet in *F, G, H, I, K*; the angles at these points are together equal to two right angles.

For since *BCG* is the exterior angle of the triangle *FCI*, it is equal to the angles at *F* and *I*. For the same reason the angle *CBG* is equal to the angles at *K* and *H*; and ∴ the angles at *F, G, H, I, K* are equal to the three angles of the triangle *BCG*, *i. e.* to two right angles.

* * *

(25.) *If the sides of an equilateral and equiangular hexagon be produced to meet; the angles formed by these lines are together equal to four right angles.*



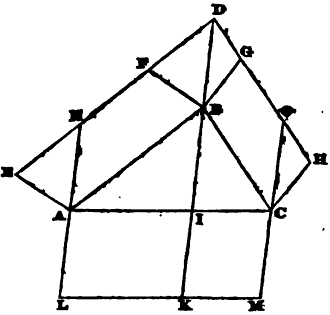
Let *ABCDEF* be an equilateral and equiangular hexagon; and let the sides be produced to meet in *G, H, I, K, L, M*; the angles at these points are together equal to four right angles.

For *GLI* being a triangle, the angles at *G, I, L* are equal to two right angles; and for the same reason, the angles at *H, K, M* are equal to two right angles; ∴ the six angles are equal to four right angles.

* * *

(26.) *The area of any two parallelograms described on the two sides of a triangle is equal to that of a paral-*

lelogram on the base, whose side is equal and parallel to the line drawn from the vertex of the triangle to the intersection of the two sides of the former parallelograms produced to meet.

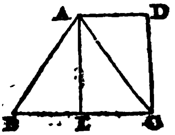


Let BE and CG be parallelograms described on the sides AB, BC of the triangle ABC; and let EF, HG be produced to meet in D. Join DB; produce it, and make $IK = DB$; through A draw AL equal and parallel to IK; and complete the parallelogram AM. AM is equal to AF and CG together.

Produce LA, MC to N and O; since ND is parallel to AB, and AN to BD, $NABD$ is a parallelogram, and equal to EB, which is on the same base, and between the same parallels. It is also equal to AK; because they are upon equal bases DB, IK, and between the same parallels; $\therefore AK = EB$. In the same manner $IM = BH$, $\therefore AM$ is equal to AF and CG together.

* * *

(27.) *The perimeter of an isosceles triangle is greater than the perimeter of a rectangular parallelogram, which is of the same altitude with, and equal to the given triangle.*



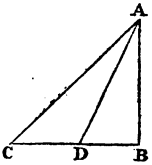
Let ABC be an isosceles triangle, whose base is BC. Draw AE perpendicular to BC, and $\therefore$ bisecting it; and draw AD, CD parallel respectively to BC, *AE*; then *DE* is a rectangular parallelogram of the same altitude with, and equal to the triangle *ABC* (Eucl. i. 42.). The perimeter of *ABC* is greater than that of *DE*.

Because $AB = AC$, and $BE = EC$, the perimeter of *ABC* is double of *AC* and *EC* together; also the perimeter of *DE* is double of *AE* and *EC* together. But since *AEC* is a right angle, *AC* is greater than *AE*; and ∴ the perimeter of *ABC* greater than that of *DE*.

* * *

(28.) *If from one of the acute angles of a right-angled triangle, a line be drawn to the opposite side; the squares of that side and the line so drawn are together equal to the squares of the segment adjacent to the right angle and of the hypothenuse.*

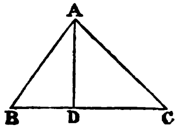
Let *ABC* be a right-angled triangle, and from *A* let *AD* be drawn to the opposite side; the squares of *AD* and *BC* are together equal to the squares of *AC* and *BD*.



For the squares of *AD* and *BC* together are equal to the squares of *AB*, *BD* and *BC*, *i.e.* to the squares of *AC* and *BD*; since the squares of *AB* and *BC* are equal to the square *AC*.

* * *

(29.) *In any triangle if a line be drawn from the vertex at right angles to the base, the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.*

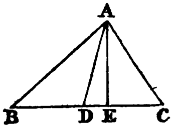


From *A* the vertex of the triangle *ABC*, let *AD* be drawn perpendicular to the base; the difference of the squares of *AB*, *AC* is equal to the difference of the squares of *BD*, *DC*.

For since *ABD* is a right-angled triangle, the square of *AB* is equal to the squares of *AD*, *BD*; and since *ADC* is a right-angled triangle, the square of *AC* is equal to the squares of *AD*, *DC*; whence the difference of the squares of *AC* and *AB* is equal to the difference of the squares of *CD* and *DB*.

* * *

(30.) *In any triangle, if a line be drawn from the vertex bisecting the base; the sum of the squares of the two sides of the triangle is double the sum of the squares of the bisecting line and of half the base.*



From the vertex *A* of the triangle *ABC*, let *AD* be drawn to the point of bisection of the base; the squares of *AB*, *AC*, are together double the squares of *AD*, *DB*.

From *A* draw *AE* perpendicular to *BC*;

Then (Eucl. ii. 12.) $AB^2 = AD^2 + DB^2 + 2BD \times DE$,

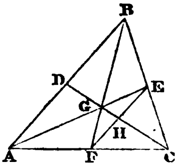
and (Eucl. ii. 13.) $AC^2 = AD^2 + DC^2 - 2CD \times DE$

$$= AD^2 + DB^2 - 2BD \times DE,$$

whence $AB^2 + AC^2 = 2AD^2 + 2DB^2$.

* * *

(31.) *If from the three angles of a triangle lines be drawn to the points of bisection of the opposite sides; the squares of the distances between the angles and the* *common intersection are together one third of the squares of the sides of the triangle.*



From the angles of the triangle ABC, let lines be drawn to the middle points of the opposite sides, intersecting each other in G; the sum of the squares of AG, GB, GC is one third of the sum of the squares of AB, BC, CA.

Join EF. Then $AB^2 + AC^2 = 2AE^2 + 2EB^2$,

$$AB^2 + BC^2 = 2AF^2 + 2FB^2,$$

$$AC^2 + BC^2 = 2AD^2 + 2DC^2,$$

$$\therefore AB^2 + BC^2 + CA^2 = AE^2 + BF^2 + CD^2 + AF^2 + [EB^2 + AD^2,$$

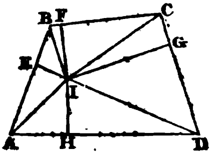
Now the sum of the squares of AF, EB, AD is equal to one fourth of the sum of the squares of AB, BC, CA; whence three fourths of the sum of the squares of AB, BC, CA will be equal to the sum of the squares of AE, BF, CD; or three times the sum of the squares of AB, BC, CA is equal to four times the sum of the squares of AE, BF, CD.

Now $BG : GF :: BA : EF :: BC : CE :: 2 : 1$,

$\therefore BG : BF :: 2 : 3$, and $BG^2 : BF^2 :: 4 : 9$, whence $4BF^2 = 9BG^2$. And the same being true of each of the rest, three times the sum of the squares of AB, BC, CA, is equal to nine times the sum of the squares of AG, BG, CG; $\therefore$ the sum of the squares of AB, BC, CA is three times the sum of the squares of AG, BG, CG.

Cor. If from the angles of a triangle lines be drawn to the points of bisection of the opposite sides, the squares of those lines together are to the squares of the sides of the triangle as 3 : 4.

(32.) *If from any point within or without any rectilineal figure, perpendiculars be let fall on every side; the sum of the squares of the alternate segments made by them will be equal.*



Let $ABCD$ be any quadrilateral figure (the demonstration being the same whatever be the number of sides). From any point I let perpendiculars IE, IF, IG, IH be drawn; $AE^2 + BF^2 + GC^2 + DH^2 = EB^2 + FC^2 + GD^2 + AH^2$.

From I draw lines to each of the angles;

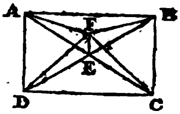
$$\begin{aligned} \text{then } AE^2 + EI^2 &= (AI^2 =)\ AH^2 + HI^2, \\ BF^2 + FI^2 &= (BI^2 =)\ BE^2 + EI^2, \\ CG^2 + GI^2 &= (CI^2 =)\ CF^2 + FI^2, \\ DH^2 + HI^2 &= (DI^2 =)\ DG^2 + GI^2, \end{aligned}$$

whence,

$$AE^2 + BF^2 + CG^2 + DH^2 = EB^2 + FC^2 + GD^2 + HA^2.$$

* * *

(33.) *If from any point within a rectangular parallelogram lines be drawn to the angular points; the sums of the squares of those which are drawn to the opposite angles are equal.*



Let $ABCD$ be a rectangular parallelogram, and F any point within it; join FA, FB, FC, FD; the squares of FA and FC are together equal to the squares of FB and FD.

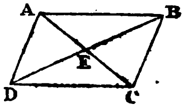
Draw the diagonals AC, BD; and join FE. Because the triangles ADC, BDC are similar and equal, $AC = BD$; and $\therefore$ their halves, AE and DE, are equal.

Now (iv. 30.) $FD^2+FB^2=2DE^2+2EF^2$,

$$=2AE^2+2EF^2=AF^2+FC^2.$$

* * *

(34.) *The squares of the diagonals of a parallelogram are together equal to the squares of the four sides.*



Let $ABCD$ be a parallelogram, whose diagonals are AC, BD; the squares of AC, BD are together equal to the squares of AB, BC, CD, DA.

Since DB is bisected by AC,

$$2AE^2+2ED^2=AD^2+AB^2,$$

and for the same reason,

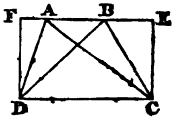
$$2CE^2+2ED^2=CD^2+CB^2,$$

$$\therefore\ 4AE^2+4ED^2=AD^2+AB^2+CB^2+CD^2,$$

$$i.e.\ AC^2+BD^2=AD^2+AB^2+CB^2+CD^2.$$

* * *

(35.) *If two sides of a trapezium be parallel to each other; the squares of its diagonals are together equal to the squares of its two sides which are not parallel and twice the rectangle contained by its parallel sides.*



Let the sides AB, DC of the trapezium $ABCD$ be parallel; draw the diagonals AC, BD; the squares of AC and BD, are together equal to the squares of AD and BC, and twice the rectangle AB, DC.

Let fall the perpendiculars CE, DF.

Then (Eucl. ii. 12.), $DB^2=DA^2+AB^2+2AB\times AF$,

and $AC^2=CB^2+AB^2+2AB\times BE$,

whence,

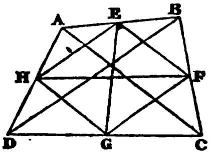
$$AC^2 + DB^2 = AD^2 + CB^2 + 2AB^2 + 2AB \times BE + 2AB \times AF.$$

Now (Eucl. ii. 1.),

$$AB \times FE = AB \times FA + AB \times AB + AB \times BE,$$

$$\therefore AC^2 + DB^2 = AD^2 + CB^2 + 2\,AB \times DC.$$

(36.) *The squares of the diagonals of a trapezium are together double the squares of the two lines joining the bisections of the opposite sides.*



Let $ABCD$ be a trapezium, whose sides are bisected in E, F, G, H. Join EG, FH; and draw the diagonals AC, BD. The squares of AC, BD are together double of the squares of EG, FH.

Join EF, FG, GH, HE. Then (iv. 14.) $EFGH$ is a parallelogram, and BD is double of EH;

$$\therefore BD^2 = 4\,EH^2 = 2\,EH^2 + 2\,FG^2,$$

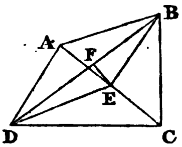
and for the same reason $AC^2 = 2\,EF^2 + 2HG^2$,

$$\therefore AC^2 + BD^2 = 2\,EF^2 + 2\,FG^2 + 2\,GH^2 + 2\,HE^2,$$

$$= 2\,EG^2 + 2\,HF^2. \text{ (iv. 34.)}$$

(37.) *The squares of the diagonals of a trapezium are together less than the squares of the four sides, by four times the square of the line joining the points of bisection of the diagonals.*

Let $ABCD$ be a trapezium whose diagonals AC, BD are bisected in E, F; join EF; the squares of AC,



BD are less than the squares of the four sides by four times the square of *EF*.

Since *BE* bisects *AC* the base of the triangle *ABC*,

$$AB^2+BC^2=2AE^2+2EB^2;$$

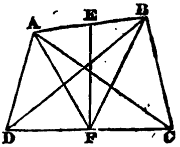
and for a similar reason,

$$AD^2+DC^2=2AE^2+2ED^2;$$

$$\begin{aligned}\therefore AB^2+BC^2+CD^2+DA^2 &= 4AE^2+2EB^2+2ED^2\\ &= AC^2+2EB^2+2ED^2\\ &= AC^2+4BF^2+4FE^2\\ &= AC^2+BD^2+4FE^2.\end{aligned}$$

* * *

(38.) *In any trapezium, if two opposite sides be bisected; the sum of the squares of the two other sides, together with the squares of the diagonals, is equal to the sum of the squares of the bisected sides together with four times the square of the line joining those points of bisection.*



Let *AB*, *DC*, two opposite sides of the trapezium *ABCD*, be bisected in *E*, and *F*; join *EF*; and draw the diagonals *AC*, *BD*. The squares of *AD*, *BC*, *AC*, *BD* are equal to the squares of *AB*, *DC*, and four times the square of *EF*.

Join *AF*, *BF*. Since *AF* bisects *DC* the base of the triangle *ADC*,

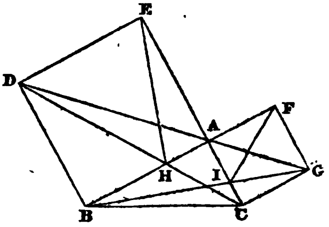
$$AD^2+AC^2=2DF^2+2FA^2;$$

and in the same manner,

$$BC^2+BD^2=2DF^2+2FB^2;$$

whence $AD^2+BC^2+AC^2+BD^2=4DF^2+2FA^2+2FB^2$
$=DC^2+2FA^2+2FB^2=DC^2+4AE^2+4EF^2$
$=DC^2+AB^2+4EF^2$.

(39.) *If squares be described on the sides of a right-angled triangle; each of the lines joining the acute angles and the opposite angle of the square, will cut off from the triangle an obtuse-angled triangle, which will be equal to that cut off from the square by a line drawn from the intersection with the side to that angle of the square which is opposite to it.*

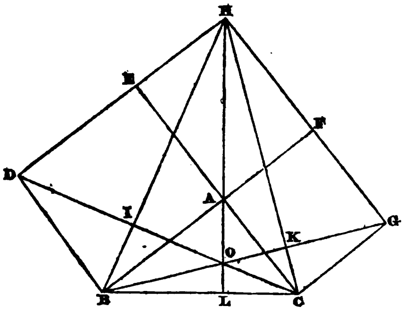


From the angles *B*, *C* of the right-angled triangle *BAC*, let lines *BG*, *CD* be drawn to the angles of the squares described upon the sides, and from the intersections *H* and *I* let *HE*, *IF* be drawn to the opposite angles of the squares; the triangle $BIC=AIF$, and $CHB=AHE$.

Join *AG*, *AD*. Then (Eucl. i. 37.) the triangle $AFI=AIG$; to each of which add *ABI*, ∴ the triangle $BIF=BAG=BCA$ (Eucl. i. 37.) From each of these equals take away the triangle *BIA*, and $BIC=AIF$. In the same manner it may be shewn that $CHB=AHE$.

(40.) *If squares be described on the two sides of a right-angled triangle; the lines joining each of the acute angles of the triangle and the opposite angle of the square will meet the perpendicular drawn from the right angle upon the hypothenuse, in the same point.*

Let BE, CF be squares described on the sides BA, AC containing the right angle. Join DC, BG; they

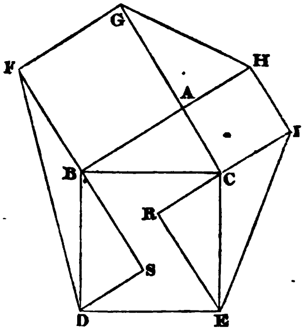


intersect AL, which is perpendicular to BC, in the same point O.

Produce DE, GF, to meet in H. Join HA, HB, HC. Let BH, CH respectively meet DC, BG in I and K. Since $EH = AF = AC$, and $EA = AB$, and the angles HEA, BAC are right angles, the triangles HEA, BAC are equal, and the angle $EHA = BCA = BAL$, *i. e.* since EH and BA are parallel, HAL is a straight line, or LA produced passes through H, and HL is perpendicular to BC. Again, since $AC = CG$, $AH = BC$, and the angle $HAC = BCG$, $\therefore$ the triangles HAC, BCG are equal; $\therefore$ the angle $CBK = CHL$; but $BCK = HCL$; $\therefore$ $BKC = HLC$, *i. e.* is a right angle, and BK is perpendicular to HC. In the same manner it may be shewn that CI is perpendicular to BH. Hence $\therefore$ HL, CI, BK are perpendicular to the sides of the triangle HBC, and $\therefore$ they intersect each other in the same point.

(41.) *If squares be described on the three sides of a right-angled triangle, and the extremities of the ad-*

jacent sides be joined; the triangles so formed are equal to the given triangle and to each other.

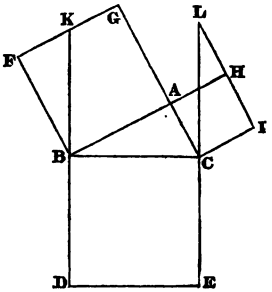


On the sides of the right-angled triangle ABC let squares be described, and join GH, FD, IE. The triangles AGH, BFD, ECI are equal to ABC, and to each other.

It is evident that $AGH = ABC$. Produce FB, and from D draw DS perpendicular to it. Since ABS and CBD are right angles, $\therefore$ the angles ABC, SBD are equal; and BAC, BSD are also right angles, and $BC = BD$, $\therefore$ $DS = AC$. And the triangles ABC, FBD being upon equal bases AB, FB are as their altitudes AC, DS (Eucl. vi. 1.); and $\therefore$ are equal. In the same manner if IC be produced, and ER drawn perpendicular to it, it may be shewn that ER is equal to AB, and the triangle ECI to ABC. And since each of the triangles is equal to ABC, they are equal to one another.

(42.) *If the sides of the square described upon the hypothenuse of a right-angled triangle be produced to meet the sides (produced if necessary) of the squares described upon the legs; they will cut off triangles equiangular and equal to the given triangle.*

Let DB, EC, the sides of the square described on BC the hypothenuse of the right-angled triangle ABC, be produced to meet the sides of the squares described upon *BA*, *AC*, in *K* and *L*; the triangles *BFK*, *CIL* cut off by them are equal and equiangular to *ABC*.

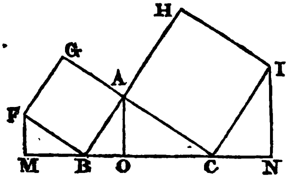


Since *FBA* and *KBC* are right angles, the angles *FBK*, *ABC* are equal; also the angles at *F* and *A* are right angles, and *FB*=*BA*, ∴ *FK*=*AC*, and the triangles *FKB*, *ABC* are equiangular and equal.

In like manner it may be proved that the triangles *ABC*, *LCI* are equiangular and equal.

* * *

(43.) *If from the angular points of the squares described upon the sides of a right-angled triangle perpendiculars be let fall upon the hypothenuse produced; they will cut off equal segments; and the perpendiculars will together be equal to the hypothenuse.*



Let *FM*, *IN* be drawn from the angles *F*, *I* of the squares described upon *BA*, *AC*, perpendicular to *BC* the hypothenuse produced; *MB* will be equal to *NC*; and *FM*, *IN* together equal to *BC*.

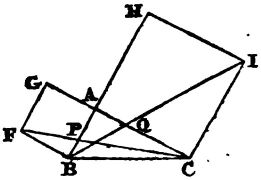
From *A* draw *AO* perpendicular to *BC*. Since *FBA* is a right angle, the angles *FBM* and *ABO* together are equal to *FBM* and *BFM*, ∴ *ABO* is equal to *BFM*; and the angles at *M* and *O* are right angles, and *AB*= *BF*, ∴ *BM*=*AO*, and *FM*=*BO*. In the same manner it may be shewn that *CN*=*AO*, and *IN*=*CO*; ∴ *MB*

$=NC$, and FM and IN together are equal to BO and CO together, *i. e.* to BC.

Cor. The triangles FBM, ICN are together equal to ABC.

* * *

(44.) *If on the two sides of a right-angled triangle squares be described, the lines joining the acute angles of the triangle and the opposite angles of the squares will cut off equal segments from the sides; and each of these equal segments will be a mean proportional between the remaining segments.*



On AB, AC the sides of the right-angled triangle BAC, let squares be described, and BI, CF joined; the segments AP, AQ are equal, and each of them is a mean proportional between BP and CQ.

Since AQ is parallel to HI, and AP to FG,

$$BH : HI :: BA : AQ,$$

$$\text{and } (CA=)\,HI : CG :: AP : (FG=)\,AB,$$

$$\therefore BH : CG :: AP : AQ;$$

and BH being equal to CG, $AP=AQ$.

Again, the triangles BPF, ACP being similar, as also ABQ, ICQ,

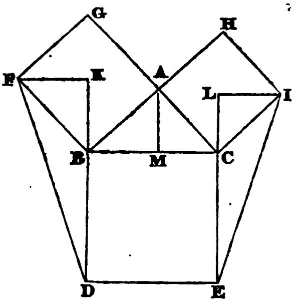
$$BP : (BF=)\,AB :: AP : AC,$$

$$\text{and } BA : AQ :: (IC=)\,AC : CQ,$$

$$\therefore \textit{ex æquo } BP : (AQ=)\,AP :: AP : CQ.$$

* * *

(45.) *If squares be described on the hypothenuse and sides of a right-angled triangle, and the extremities of* *the sides of the former and the adjacent sides of the others be joined; the sum of the squares of the lines joining them will be equal to five times the square of the hypothenuse.*



Let squares be described on the three sides of the right-angled triangle ABC; join DF; EI; the squares of DF and EI together are equal to five times the square of BC.

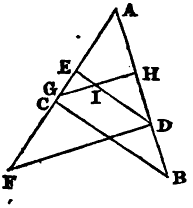
Draw FK, IL perpendicular to DB, EC produced, and AM to BC. The angle FBK is equal to ABC, and the angle at K to the right angle AMB, and $FB = BA$, $\therefore BK = BM$. In the same way, $CL = CM$.

Now (Eucl. ii. 12.)
$$FD^2 = DB^2 + BF^2 + 2DB \times BK$$
$$= BC^2 + BA^2 + 2BC \times BM,$$
$$\text{and } EI^2 = BC^2 + CA^2 + 2BC \times CM,$$
$$\therefore FD^2 + EI^2 = 2BC^2 + BA^2 + AC^2 + 2BC \times BM + 2BC \times CM$$
$$= 2BC^2 + BC^2 + 2BC^2 = 5BC^2.$$

* * *

(46.) *If a line be drawn parallel to the base of a triangle, and terminated in the sides; to draw a line cutting it, and terminated also by the sides, so that the rectangle contained by their segments may be equal.*

Let ED be parallel to CB the base of the triangle ABC; from D draw DF, making with AC (produced if



necessary) the angle DFE equal to ABC, and draw any line GH parallel to FD, cutting ED in I; the rectangle EI, ID is equal to the rectangle GI, IH.

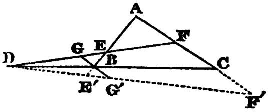
For the angle $AGH=AFD=ABC=ADE$, and the vertical angles at I are equal, ∴ the triangles GEI, HID are equiangular;

$$\text{and } HI : ID :: IE : IG,$$

∴ the rectangle EI, ID is equal to the rectangle HI, IG.

* * *

(47.) *If the sides, or sides produced, of a triangle be cut by any line; the solids formed by the segments which have not a common extremity are equal.*



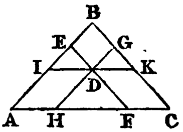
Let ABC be a triangle having the sides (produced if necessary) cut by the line DEF; then $AF \times CD \times BE = AE \times DB \times CF$.

Draw BG parallel to AC; the triangles AEF, BEG will be similar, as also CDF, BDG;

$$\therefore AF : AE :: BG : BE,$$
$$\text{and } CD : CF :: BD : BG,$$
$$\therefore AF \times CD : AE \times CF :: BD : BE,$$
$$\therefore AF \times CD \times BE = AE \times DB \times CF.$$

* * *

(48.) *If through any point within a triangle, three lines be drawn parallel to the sides; the solids formed by the alternate segments of these lines are equal.*



Through any point D within the triangle ABC, let HG, EF, IK, be drawn parallel to the sides; then $ID \times DG \times DF = ED \times DK \times DH$.

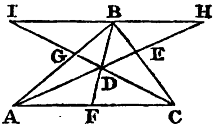
Since the lines are drawn parallel to the sides, the triangles IED, GDK, HDF are similar to ABC, and to one another;

$$\therefore ID : DE :: AC : CB$$
$$GD : DK :: AB : AC$$
$$DF : DH :: BC : AB,$$

whence $ID \times DG \times DF : DE \times DK \times DH :: AC \times AB \times BC : BC \times AC \times AB$,

i. e. in a ratio of equality.

(49.) *If through any point within a triangle lines be drawn from the angles to cut the opposite sides; the segments of any one side will be to each other in the ratio compounded of the ratios of the segments of the other sides.*



Through any point D within the triangle ABC, let lines AE, BF, CG be drawn from the angles to the opposite sides; the segments of any one of them as AC, will be in the ratio compounded of the ratios $AG : GB$, and $BE : EC$.

Draw IBH parallel to AC, meeting AE and CG produced in H and I. Then the triangles GCA, GBI, and EAC, EBH, as also ADF, BDH, and FDC, IDB, are respectively equiangular,

whence $BH : AC :: BE : CE$,
and $AC : BI :: AG : BG$,

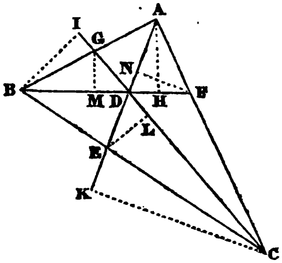
$\therefore BH : BI :: AG \times BE : GB \times CE.$

But $BH : BI :: AF : FC,$

$\therefore AF : FC :: AG \times BE : GB \times CE.$

* * *

(50.) *If from each of the angles of any triangle, a line be drawn through any point within the triangle, to the opposite side; the solid contained by the segments thereof, intercepted between the angles and the point, will have to the solid contained by the three remaining segments, the same ratio that the solid contained by the three sides of the triangle, has to either of the (equal) solids contained by the alternate segments of the sides.*



Let ABC be the given triangle, and through any point D within it, let AE, BF, CG be drawn from the angles to the opposite sides; then will $AD \times DB \times DC : ED \times DF \times DG :: AB \times BC \times CA : AF \times CE \times BG.$

Let fall the perpendiculars AH, BI, CK; EL, GM, FN.

Since EL is parallel to BI, $CB : CE :: BI : EL$, and GM being parallel to AH, $BA : BG :: AH : GM$, also FN and CK being parallel, $AC : AF :: CK : FN$,

$\therefore AC \times AB \times BC : CE \times BG \times AF :: BI \times AH \times CK :$
[$EL \times GM \times FN.$

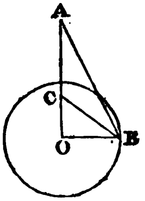
Again, since EL is perpendicular to DC, and CK to DK, the triangles DEL, DCK are equiangular,

and $\therefore DC : DE :: CK : EL.$

In the same manner, $DB : DG :: BI : GM$,
and $DA : DF :: AH : FN$,
$\therefore DA \times DB \times DC : DE \times DF \times DG :: BI \times AH \times CK : EL \times GM \times FN :: AB \times BC \times CA : AF \times CE \times BG$.

SECT. V.

(1.) *A straight line of given length being drawn from the centre at right angles to the plane of a circle; to determine that point in it, which is equally distant from the upper end of the line and the circumference of the circle.*



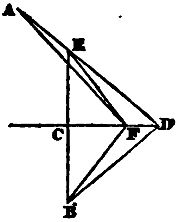
From O the centre of the circle, let OA be drawn at right angles to its plane; draw OB perpendicular to OA; join AB, and make the angle ABC equal to BAC. C is the point required.

Since the angle $ABC = CAB$, $\therefore AC = CB$.

* * *

(2.) *To determine a point in a line given in position, to which lines drawn from two given points may have the greatest difference possible.*

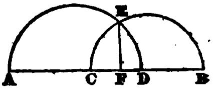
Let A and B be the given points, and CD the line given in position. Let fall the perpendicular BC, and



produce it, so that CE may be equal to CB; join AE, and produce it to meet CD in D. Join BD. D is the point required.

For $DE = DB$; and $\therefore AE$ is equal to the difference between AD and DB. If then any other point F be taken, $BF = EF$; and the difference between AF and BF is equal to the difference between AF and EF, which is less than AE (iii. 1.). The same may be proved for every other point in CD.

(3.) *A straight line being divided in two given points; to determine a third point such that its distances from the extremities may be proportional to its distances from the given points.*



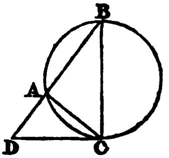
Let AB be the given line, divided in C and D. On AD and CB let semicircles be described intersecting in E. From E let fall the perpendicular EF; F is the point required.

For (Eucl. vi. 8. Cor.) $AF : FE :: FE : FD$,

and $FE : FB :: FC : FE$,

$\therefore AF : FB :: FC : FD$.

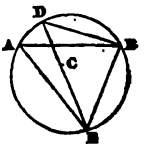
(4.) *In a straight line given in position, to determine a point, at which two straight lines, drawn from given points on the same side, will contain the greatest angle.*



Let *A* and *B* be the given points, and *CD* the given line. Join *BA*, and produce it to meet *CD* in *D*. Take *DC* a mean proportional between *DA* and *DB*. *C* is the point required.

Join *AC*, *BC*; and about the triangle *ABC* describe a circle; *DC* is a tangent at the point *C* (Eucl. iii. 37.), and ∴ the angle is the greatest (ii. 62.).

(5.) *To determine the position of a point, at which lines drawn from three given points, shall make with each other angles equal to given angles.*

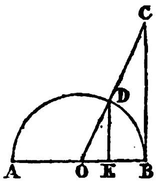


Let *A*, *B*, *C* be the three given points; join *AB*, and on it describe a segment of a circle containing an angle equal to that which the lines from *A* and *B* are to include. Complete the circle, and make the angle *ABD* equal to that which the lines from *A* and *C* are to include. Join *DC*, and produce it to the circumference in *E*. *E* is the point required.

Join *AE*, *BE*. Then the angle *AEC*=*ABD*, and *AEB* is of the given magnitude, by construction.

(6.) *To divide a straight line into two parts such, that the rectangle contained by them may be equal to the square of their difference.*

Let *AB* be the given line; upon it describe a semicircle *ADB*. From *B* draw *BC* at right angles and

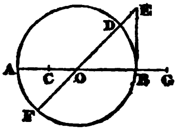


equal to *AB*. Take *O* the centre, and join *OC*; and from *D* draw *DE* perpendicular to *AB*; *AB* is divided in the point *E*, as was required.

Since *BC* is double of *BO*, *DE* is double of *OE* (Eucl. vi. 2.), and *OE* being half the difference between *AE* and *EB*, *DE* is equal to the difference. Also (Eucl. vi. 13.) the rectangle *AE*, *EB* is equal to the square of *DE*.

* * *

(7.) *If a straight line be divided into any two parts; to produce it, so that the rectangle contained by the whole line so produced, and the part produced may be equal to the rectangle contained by the given line and one segment.*



Let *AB* be the given line divided into two parts in the point *C*. On *AB* as diameter describe a circle *ADB*. From *B* draw *BE* at right angles to *AB*, and ∴ a tangent to the circle; and make *BE* a mean proportional between *AB* and *AC*. Take *O* the centre; join *EO*, and produce it to *F*. Produce *AB* to *G*, making *BG* equal to *ED*. Then will the rectangle *AG*, *GB* be equal to the rectangle *BA*, *AC*.

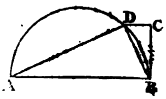
Since $DE = BG$, the rectangle *BG*, *GA* is equal to the rectangle *DE*, *EF*, *i. e.* to the square of *EB*, or to the rectangle *AB*, *AC*, by construction.

Cor. 1. If it be required to produce the line, so that the rectangle contained by the whole line produced and the part produced, may be equal to the rectangle contained by two given lines; find *BE* a mean proportional between the two given lines, and proceed as in the proposition.

Cor. 2. If it be required to produce the line, so that the rectangle contained by the whole line produced and the part produced, may be equal to a given square; take *BE* equal to a side of the square, and proceed as in the proposition.

* * *

(8.) *To determine two lines such that the sum of their squares may be equal to a given square, and their rectangle equal to a given rectangle.*



Let *AB* be equal to a side of the given square. Upon it describe a semicircle *ADB*; and from *B* draw *BC* perpendicular to *AB*, and equal to a fourth proportional to *AB* and the sides of the given rectangle. From *C* draw *CD* parallel to *BA*. Join *AD*, *DB*; they are the lines required.

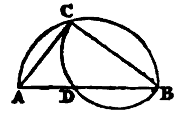
Since *CB* touches the circle at *B*, the angle *CBD* is equal to *DAB*, and the angles *DCB*, *ADB* are right angles; ∴ the triangles *DCB*, *ADB* are equiangular,

$$\text{and } AB : AD :: DB : BC,$$

whence the rectangle *AD*, *DB* is equal to the rectangle *AB*, *BC*, *i. e.* to the given rectangle. Also the squares of *AD*, *DB* are equal to the square of *AB*, *i. e.* to the given square.

* * *

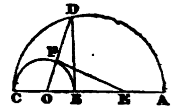
(9.) *To divide a straight line into two parts, so that* *the rectangle contained by the whole and one of the parts may be equal to the square of a given line, which is less than the line to be divided.*



Let AB be the given line to be divided. Upon it describe a semicircle, in which place the line $AC =$ to the given line. Join CB; and on it describe a semicircle CDB, cutting AB in D; D is the point required.

Since the angle ACB is in a semicircle, it is a right angle, ∴ AC touches the circle CDB (Eucl. iii. 16. Cor.); whence the rectangle BA, AD is equal to the square of AC, *i. e.* to the square of the given line.

(10.) *To divide a given line into two such parts that the rectangle contained by the whole line and one of the parts may be* (m) *times the square of the other part,* m *being whole or fractional.*



Let AB be the given line, and in it produced, take $BC =$ an m^{th} part of AB. On AC describe a semicircle, and from B draw BD perpendicular to AC. Bisect CB in O; join OD, and take $OE = OD$; and AB will be divided in E, as required.

On BC describe a semicircle, cutting OD in F; join FE. Then the angle DOE being common to the triangles DOB, EOF, and DO, OB respectively equal to EO, OF, the triangles will be similar and equal, and ∴ the angle OFE equal to OBD, and ∴ a right angle; whence FE is a tangent to the circle CFB. Hence the rectangle AB, BC is equal to the square of DB, *i. e.* to the square of FE, or the rectangle CE, EB.

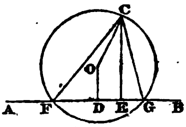
From each of these equals take away the rectangle *CB*, *BE*; and the rectangle *AE*, *CB* is equal to the square of *BE*, ∴ (*m*) times the rectangle *AE*, *CB*, *i. e.* the rectangle *AB*, *AE* is equal to (*m*) times the square of *BE*.

(11.) *To divide a given line into two such parts that the square of the one shall be equal to the rectangle contained by the other and a given line.*

Let *AB* be the given line to be divided, (see last Fig.) and *BC* the other given line. Let them be placed so as to be in the same straight line. On *AC* describe a semicircle and draw the lines, as in the last proposition; and *E* is the point required.

For the rectangle *AE*, *CB* is equal to the square of *BE*.

(12.) *A straight line being given in magnitude and position; to draw to it from a given point, two lines, whose rectangle shall be equal to a given rectangle, and which shall cut off equal segments from the given line.*

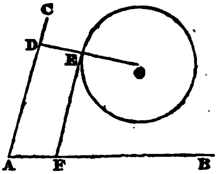


Let *AB* be the given line, and *C* the given point. Bisect *AB* in *D*, and from *D* draw *DO* at right angles to *AB*, and let fall the perpendicular *CE*. With the centre *C*, and radius equal to a fourth proportional to 2 *CE* and the sides of the given rectangle, describe a circle cutting *DO* in *O*. Join *OC*; and with the centre *O*, and radius OC, describe a circle CFG, cutting AB in F and G; join CF, CG; they are the lines required.

For (Eucl. vi. C.) the rectangle CF, CG is equal to the rectangle contained by $2\,CO$ and CE, *i. e.* to the given rectangle. And since $AD=DB$, and $FD=DG$, $\therefore AF=GB$.

* * *

(13.) *To draw a straight line which shall touch a given circle, and make with a given line, an angle equal to a given angle.*



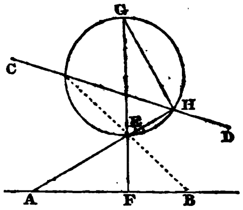
Let AB be the given line, and O the centre of the given circle. From any point A in the given line, draw AC making with it an angle equal to the given angle; from O draw OD perpendicular to AC, and through the point E where it meets the circle, draw EF parallel to DA; EF is the line required.

For being parallel to AC it is perpendicular to OD, and $\therefore$ a tangent to the circle; and the angle $EFB = DAB =$ the given angle.

* * *

(14.) *Through a given point to draw a line terminating in two lines given in position, so that the rectangle contained by the two parts may be equal to a given rectangle.*

Let AB, CD be the lines given in position, E the given point; from E draw EF perpendicular to AB, and



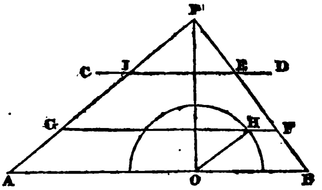
and produce *FE* to *G*, so that the rectangle *FE*, *EG* may be equal to the given rectangle. On *EG* describe a circle, cutting *CD* in *H*. Join *HE*, and produce it to *A*; *AH* is the line required.

Join *GH*. The angle *GEH* is equal to *AEF*, and the angles *GHE*, *AFE* are right angles, ∴ the triangles *GEH*, *AEF* are equiangular, and

$$EH : EG :: EF : EA,$$

whence the rectangle *AE*, *EH* is equal to the rectangle *EG*, *EF*, *i.e.* to the given rectangle.

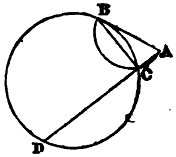
(15.) *From a given point to draw a line cutting two given parallel lines, so that the difference of its segments may be equal to a given line.*



Let *AB*, *CD* be the given parallels, and *P* the given point. From *P* draw any line *PB*, meeting the given lines in *B* and *E*. Make *EF* = *EP*, and draw *FG* parallel to *AB*. With any point *O* as centre, and radius equal to the given line, describe a circle cutting *GF* in *H*. Join *OH*, and draw *PGA* parallel to it. *PGA* will be the line required.

Since *PE* is equal to *EF*, ∴ (Eucl. vi. 2.) $PI = IG$; and *AG* is equal to the difference of *AI* and *IP*, the segments of *PA*; and $AG = OH =$ the given line.

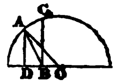
(16.) *From a given point without a circle, to draw a straight line cutting the circle, so that the rectangle contained by the part of it without and the part within the circle shall be equal to a given square.*



Let *A* be the given point, and *BCD* the given circle. From *A* draw *AB* touching the circle; and on it as a diameter describe a semicircle, cutting the given circle in *C*. Join *AC*, and produce it to *D*. Now if the side of the given square be equal to *BC*, the problem is possible.

For the rectangle *AC, AD* is equal to the square of *AB*, *i. e.* to the squares of *AC* and *BC*; take away from each the square of *AC*, ∴ the rectangle *AC, CD* is equal to the square of *BC*.

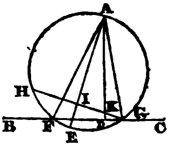
(17.) *From a given point in the circumference of a semicircle, to draw a straight line meeting the diameter, so that the difference between the squares of this line and a perpendicular to the diameter from the point of intersection may be equal to a given rectangle.*



Let *A* be the given point in the circumference of the semicircle; from it draw *AD* perpendicular to the diameter. Take *O* the centre, and divide *DO* in *B*, so that the rectangle contained by 2 *OB, BD* may be equal to the given rectangle. Join *AB*; and draw *BC* perpendicular to *DB*. *AB, BC* are the lines required.

For (Eucl. ii. 12.) the square of *AB* together with twice the rectangle *OB, BD* is equal to the difference of the squares of *OA* and *OB*, *i. e.* to the square of *BC*; ∴ the difference between the squares of *AB* and *BC* is equal to twice the rectangle *OB*, *BD*, *i. e.* to the given rectangle.

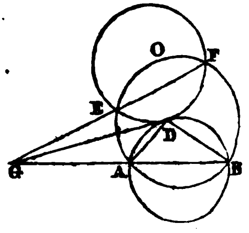
(18.) *From a given point to draw two lines to a third given in position, so that the rectangle contained by those lines may be equal to a given rectangle, and the difference of the angles which they make with that part of the third which is intercepted between them may be equal to a given angle.*



Let *A* be the given point, and *BC* the line given in position. From *A* draw *AD* perpendicular to *BC*; make the angle *DAE* equal to the given angle; and produce *AE*, till the rectangle *DA*, *AE*, is equal to the given rectangle. On *AE* as a diameter describe the circle *AFG*, cutting *BC* in *F* and *G*. Join *AF*, *AG*; they are the lines required.

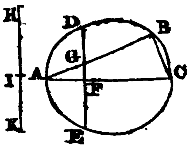
Draw *GH* perpendicular to the diameter *AE*; then the arc *HA* is equal to the arc *AG*, and the angle *AGH* to *AFG*; ∴ the angle *HGF* is equal to the difference of the angles *AGF*, *AFG*. Now the right-angled triangles *AIK*, *KDG* have the angles at *K* equal, ∴ the angle *KAI*=*KGD*; but *KAI* was made equal to the given angle; ∴ the difference of the angles *AFG*, *AGF* is equal to the given angle. And (Eucl. vi. C.) the rectangle *AF*, *AG* is equal to the rectangle *DA*, *AE*, *i. e.* to the given rectangle.

(19.) *Two points being given without a given circle; to determine a point in the circumference, from which lines drawn to the two given points shall contain the greatest possible angle.*



Let *A* and *B* be the given points, and *EDF* the given circle whose centre is *O*. Describe a circle through *A*, *B*, *O*. Join *EF*, *BA*, and produce them to meet in *G*. From *G* draw *GD* touching the given circle in *D*. Through *D*, *A*, *B* describe another circle; then since the square of *GD* is equal to the rectangle *EG*, *GF*, *i. e.* to the rectangle *AG*, *GB*, $\therefore$ *GD* touches the circle *ABD*. Join *AD*, *DB*. *D* is the point required, as is evident from (ii. 2.)

(20.) *From the bisection of a given arc of a circle to draw a straight line such that the part of it intercepted between the chord of that and the opposite circumference shall be equal to a given straight line.*

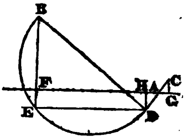


Let *DAE* be the given arc of the circle *ABC*, bisected in *A*; *AFC* the diameter, and *HI* the given straight line. Produce *HI* to *K*, so that the rectangle *HK*, *KI* may be equal to the rectangle *FA*, *AC*. From *A* place in the circle *AB* = *HK*; *AB* is the line required.

Join *BC*; then the angle *AFG* being a right angle is equal to the angle *ABC*, and the angle at *A* is common, $\therefore$ the triangles *AGF*, *ABC* are equiangular, and $AF : AG :: AB : AC$,

∴ the rectangle GA, AB is equal to the rectangle FA, AC, *i. e.* to the rectangle HK, KI. But $AB = HK$, ∴ $AG = KI$, and consequently $GB = HI$.

(21.) *To draw a straight line through a given point, so that the sum of the perpendiculars to it from two other given points may be equal to a given line.*

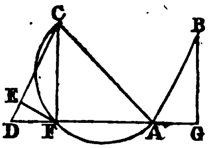


Let A, B, C be the three given points, A being that through which the line is to be drawn. Join AC, and produce it, making $AD = AC$. Join BD, and on it describe a semicircle; in which place BE equal to the given line. Join DE; and through A draw FAG parallel to DE; it is the line required.

For let fall the perpendicular CG, and draw DH parallel to BE; then the triangles ACG, AHD being similar, and $AC = AD$, ∴ $CG = HD = FE$, FD being a parallelogram; ∴ BF and CG together are equal to BE, *i. e.* to the given line; and FH being parallel to ED, BF is perpendicular to FG.

(22.) *To draw a straight line through one of three points given in position; so that the rectangle contained by the perpendiculars let fall upon it from the other two may be equal to a given square.*

Let A, B, C be the three given points, and A the point through which the line is to be drawn. Join AB,



AC; and draw *CD* parallel to *BA*, and take *CE* a third proportional to *AB* and a side of the given square. On *AC* describe a semicircle; and from *E* draw *EF* at right angles to *CD*, and meeting the semicircle in *F*. Join *AF*, and produce it; it is the line required.

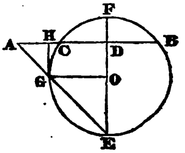
Join *CF*, which will be perpendicular to *AD*; and from *B* draw *BG* perpendicular to *AG*. Since *CE* is parallel to *BA*, and *CF* to *BG*, the triangles *ABG*, *CEF* will be similar,

$$\therefore AB : BG :: CF : CE,$$

∴ the rectangle *BG*, *CF*, is equal to the rectangle *AB*, *CE*. But since the side of the given square is, by construction, a mean proportional between *AB* and *CE*, the rectangle *AB*, *CE*, is equal to the given square; ∴ the rectangle *BG*, *CF* is equal to the given square.

* * *

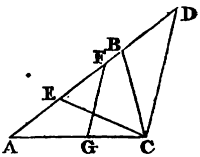
(23.) *A given straight line being divided into two parts; to cut off a part which shall be a mean proportional between the two remaining segments.*



Let *AB* be divided into two parts in the point *C*; bisect *CB* in *D*, and draw *DE* perpendicular, and equal to *AD*; and through the points *B*, *C*, *E* describe a circle; produce *ED* to *F*. Join *AE*, and bisect *EF* in *O*; and from *O* draw *OG* parallel to *AB*, meeting *AE* in *G*; and since $AD = DE$, ∴ $GO = OE$, and *G* is a point in the circumference. From *G* draw *GH* perpendicular to *AC*; *H* is the point required.

For HG, being perpendicular to AD, is perpendicular also to GO, and $\therefore$ is a tangent at G; $\therefore$ the square of HG is equal to the rectangle CH, HB. But since $AD=DE$, $\therefore AH=HG$, and consequently the square of AH is equal to the rectangle CH, HB; and AH is a mean proportional between the two remaining segments CH and HB.

(24.) *To draw a straight line making a given angle with one of the sides of a given triangle, so that the triangle cut off may be to the whole in a given ratio.*



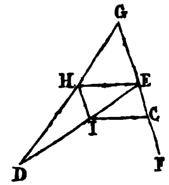
Let ABC be the given triangle; make the angle ACD equal to the given angle which the cutting line is to make with AC. Produce AB to D; and make $AE : AB$ in the ratio of the part to be cut off to the whole. Take AF a mean proportional between AE and AD; draw FG parallel to CD; FG is the line required.

Join EC. Then the triangle $ADC : AFG :: AD^2 : AF^2 :: AD : AE :: ACD : ACE$, and $\therefore AFG = ACE$.

$$\text{But } ACE : ACB :: AE : AB,$$

$\therefore AFG : ACB :: AE : AB$, *i. e.* in the given ratio.

(25.) *Between two given straight lines containing a given angle, to place a straight line of given length, and subtending that angle, so that the segment of the one of them adjacent to the angle may be to the segment of the other which is not adjacent, in the ratio of two given lines.*



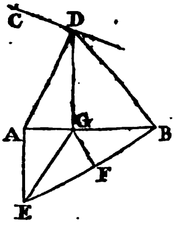
Let *ED*, *EF* be the lines given in length and position. Produce one of them *FE*, till *EG* : *ED* in the given ratio. Join *DG*; and with the centre *E*, and radius equal to the given line to be placed, describe a circle cutting *DG* in *H*; join *EH*, and draw *HI* parallel to *EF*, and *IC* parallel to *HE*. *IC* is the line required.

For (Eucl. vi. 2.) $HI : ID :: GE : ED$,

and *HC* being a parallelogram, $HI = EC$,

$\therefore EC : ID :: GE : ED$, *i. e.* in the given ratio; and $IC = EH =$ the given line.

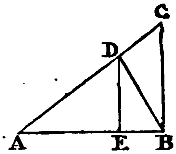
(26.) *From two given points to draw two lines to a point in a third, such that the difference of their squares may be equal to a given square.*



Let *A* and *B* be the given points; join *AB*; and from *A* draw *AE* perpendicular to it, and equal to a side of the given square. Join *BE*, and bisect it in *F*; from *F* draw the perpendicular *FG*, meeting *AB* in *G*; and from *G* draw *GD* perpendicular to *AB*, meeting *CD* in *D*; join *AD*, *DB*; these are the lines required.

Join *GE*, it is equal to *GB*. And (iv. 30.) the difference between the squares of *BD* and *AD* is equal to the difference between the squares of *BG* and *GA*, *i. e.* between the squares of *EG* and *GA*, or it is equal to the square of *AE*, *i. e.* to the given square.

(27.) *To divide a given straight line into two such parts, that the square of one may be to the excess of a given rectangle above the square of the other, in a given ratio.*

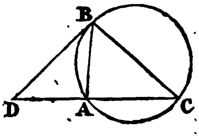


Let AB be the given straight line. From B draw BC at right angles to AB, and make $AB^2 : BC^2$ in the given ratio. Join AC. Find a mean proportional between the sides of the given rectangle; and with it as radius, and B as centre describe a circle cutting AC in D. Join BD, and draw DE parallel to BC; E is the point required.

For (Eucl. vi. 2.) $AE^2 : ED^2 :: AB^2 : BC^2$. Now the square of ED is equal to the difference of the squares of BD and BE, *i. e.* to the difference of the given rectangle and the square of BE; $\therefore$ the square of AE is to the difference between the given rectangle and the square of BE as $AB^2 : BC^2$, *i. e.* in the given ratio.

N.B. The given rectangle must not be less than the square of the perpendicular from B upon AC; and when BD is less than BC, there are two points E.

(28.) *From any angle of a triangle, not isosceles about the angle, to draw a line without the triangle to the opposite side produced, which shall be a mean proportional between the segments of the side.*



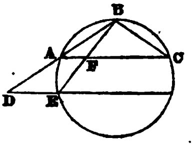
Let ABC be the triangle, and B the angle from which the mean proportional is to be drawn. About the triangle describe a circle, and to the point B

draw a tangent BD meeting the side CA produced in D. BD is a mean proportional between AD and DC.

(Eucl. iii. 36.) the rectangle AD, DC is equal to the square of DB; and $\therefore AD : DB :: DB : DC$.

* * *

(29.) *From the obtuse angle of any triangle, to draw a line within the triangle to the opposite side, which shall be a mean proportional between the segments of the side.*



Let ABC be a triangle having the obtuse angle ABC. Describe a circle about it, and produce BA to D, making $AD = AB$. From D draw DE parallel to AC, meeting the circle in E; join BE, cutting AC in F; BF will be a mean proportional between AF and FC.

For (Eucl. vi. 2.) $BF : FE :: BA : AD$,

and since $BA = AD$, $\therefore BF = FE$.

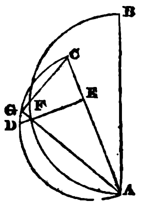
Now the rectangle AF, FC is equal to the rectangle BF, FE, *i. e.* to the square of BF;

$$\therefore AF : FB :: FB : FC.$$

* * *

(30.) *From the common extremity of the diameters of two semicircles given in magnitude and position; to draw a line meeting the circumferences, so that the rectangle contained by the two chords may be equal to a given square.*

Let AB, AC be the diameters drawn from A, and given in magnitude and position. With the centre A,



and radius equal to a side of the given square, describe a circle, cutting the lesser semicircle in *D.* Draw *DE* perpendicular to *AC,* and meeting the other semicircle in *F*. Join *AF,* and produce it to *G*; *AG* is the line required.

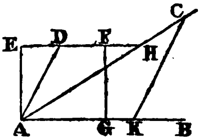
For joining *GC,* the triangles *AGC, AFE* are similar,

$$\therefore AC : AG :: AF : AE,$$

and ∴ the rectangle *FA, AG* is equal to the rectangle *CA, AE, i. e.* to the square of *AD,* which is equal to the given square.

* * *

(31.) *To draw a line parallel to a given line, which shall be terminated by two others given in position, so as to form with them a triangle equal to a given rectilineal figure.*



Let *AB, AC* be the lines given in position, *AD* the line to which it is required to draw a parallel. Describe a rectangular parallelogram *AEFG* equal to the given figure. Produce *EF* to *H*; and take *AK* a mean proportional between *DH* and 2*EF*; draw *KC* parallel to *AD*; *KC* is the line required.

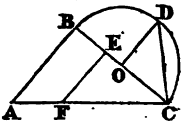
For the angles *DHA, CAK* being equal, as also *DAH, ACK,* the triangles *DAH, AKC* are equiangular, and similar; whence

$$AKC : AHD :: AK^2 : DH^2 :: 2EF : DH :: 2EF$$
$$[\times AE : DH \times AE.$$

Now the rectangle *DH, AE* is double of the triangle *AHD,* ∴ *AKC* is equal to the rectangle *EF, AE, i. e.* to the given rectilineal figure.

* * *

(32.) *To bisect a triangle by a line drawn parallel to one of its sides.*



Let *ABC* be a triangle to be bisected by a line parallel to its side *AB*. On *BC* describe a semicircle; bisect *BC* in *O*, and draw the perpendicular *OD*; join *CD*; and with *C* as centre, and radius *CD*, describe a circle cutting *CB* in *E*; draw *EF* parallel to *AB*; *EF* bisects the triangle.

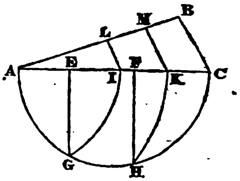
(Eucl. vi. 8.) $BC : CD :: CD : CO$,

$\therefore BC^2 : (CD^2 =) CE^2 :: BC : CO :: 2 : 1$;

but the triangles *ABC*, *FEC* are in the duplicate ratio of $BC : CE$, and $\therefore$ in the ratio of $2 : 1$, *i. e.* *EFC* is half of *ABC*, and *EF* bisects the triangle.

* * *

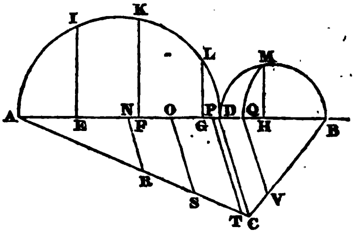
(33.) *To divide a given triangle into any number of parts having a given ratio to each other, by lines drawn parallel to one of the sides of the triangle.*



Let *ABC* be the given triangle; divide *AC* into parts *AE*, *EF*, *FC* having the same ratio to one another that the parts of the triangle are to have. On *AC* describe a semicircle, and draw the perpendiculars *EG*, *FH*; and with the centre *A*, and radii *AG*, *AH*, describe circles meeting *AC* in *I* and *K*, from which points draw *IL*, *KM* parallel to *BC*; these will divide the triangle in the ratio required.

For the triangles *ALI*, *AKM*, *ABC* are to one another in the duplicate ratio of the sides *AI*, *AK*, *AC*, *i. e.* in the ratio of the rectangles *AC*, *AE*; *AC*, *AF*; and the square of *AC*; or in the ratio of the lines *AE, AF, AC*; whence *ALI, LIKM, MKCB* are in the ratio of *AE, EF, FC, i. e.* in the given ratio.

(34.) *To divide a given triangle into any number of equal parts by lines drawn parallel to a given line.*



Let *ABC* be the given triangle; from the angle *C* draw *CD* parallel to the given line; and let it be required to divide the triangle into five equal parts. On *AD, BD* describe semicircles *AID, BMD*; divide *AB* into five equal parts in the points *E, F, G, H*; draw *EI, FK, GL, HM* perpendicular to *AB*; and make *AN, AO, AP* respectively equal to *AI, AK, AL*, and *BQ*=*BM*; and draw *NR, OS, PT, QV*, parallel to *DC*; they divide the triangle as required.

(Eucl. vi. 1.) the triangle $ABC : ADC :: AB : AD$, (Eucl. vi. 19.) $ACD : ANR :: AD^2 : AN^2 :: AD : AE$.

$\therefore$ *ex æquo*, $ABC : ANR :: AB : AE :: 5 : 1$,

i. e. ANR is one fifth of *ABC*.

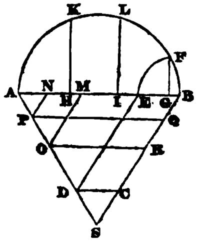
In the same manner $ABC : AOS :: 5 : 2$,

whence *NRSO* is also one fifth of *ABC*.

And by a similar manner, *OPTS* and *BQV*, may each be shewn to be one fifth of *ABC*, $\therefore$ *TPQV* will also be one fifth of *ABC*.

Cor. In nearly the same manner the triangle may be divided into any number of parts having a given ratio.

(35.) *To divide a trapezium which has two sides parallel, into any number of equal parts, by lines drawn parallel to those sides.*



Let *ABCD* be the given trapezium having the sides *AB, DC* parallel. On *AB* the longer side describe a semicircle *AFB*; from *D* draw *DE* parallel to *BC*; with the centre *B*, and radius *BE*, describe the arc *EF*, and from *F* let fall the perpendicular *FG*; and divide *AG* into the given number of equal parts, *e.g.* three, in *H* and *I*; and draw *HK, IL* at right angles to *AB*. Make *BM, BN* respectively equal to *BL, BK*; and draw *MO, NP* parallel to *BC*; and *PQ, OR* parallel to *AB*; and produce *AD, BC* to *S*.

Since *DC* = *BE* = *BF*, and *OR* = *BM* = *BL*, and *PQ* = *BN* = *BK*, the triangle *ORS* is to *DSC* in the duplicate ratio of *OR* to *CD*, or of *BL* to *BF*, *i.e.* in the ratio of *BI* : *BG*;

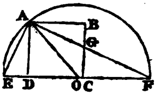
whence *ODCR* : *DSC* :: *GI* : *GB*.
In the same manner *PDCQ* : *DSC* :: *GH* : *GB*,
∴ *ODCR* : *PDCQ* :: *GI* : *GH*,
and *ODCR* : *PORQ* :: *GI* : *IH*,

i. e. in a ratio of equality.

And in a similar manner *APQB* may be shewn to be equal to *PORQ*. And so on, whatever be the number of equal parts.

Cor. In nearly the same manner, the trapezium might be divided into parts having any given ratio.

(36.) *From one of the angular points of a given square to draw a line meeting one of the opposite sides, and the other produced, in such a manner, that the exterior triangle formed thereby may have a given ratio to the square.*



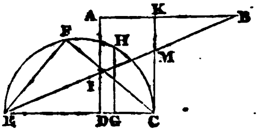
Let $ABCD$ be the given square, and $M : N$ the given ratio. From A to DC (produced if necessary) draw a line AO, such that $M : M+N :: DC : AO$. With the centre O and radius OA, describe a semicircle meeting DC produced in E and F. Join AF; which will be the line required.

Join AE. Then $M : M+N :: DC : AO :: ABCD :$ the rectangle AO, AD. Now the triangle ADE is similar to ABG, and equal to it, since $AB=AD$; $\therefore$ the trapezium $AECG$ is equal to $ABCD$; and the rectangle AO, AD is equal to the triangle AEF,

$$\text{whence } M : M+N :: AECG : AEF,$$
$$\therefore M : N :: AECG : GCF$$
$$:: ABCD : GCF.$$

* * *

(37.) *From a given point in the side produced, of a given rectangular parallelogram, to draw a line which shall cut the perpendicular sides and the other side produced, so that the trapezium cut off, which stands on the aforesaid side, may be to the triangle which stands upon the produced part of the opposite side, in a given ratio.*



Let $AKCD$ be the given rectangle, and E the given point in the side CD produced. On EC describe a semicircle, and in it place

EF=*ED*; join *FC*; and divide *EC* in *G*, so that *EG* : *GC* in the given ratio, and draw *GH* at right angles to *EC*. In *AK* produced take *BK* a fourth proportional to *EG*, *GH* and *FC*. Join *BE*; it is the line required.

For (Eucl. vi. 19.) the triangle *ECM* : *EDI* :: $EC^2 : ED^2$,

$\therefore$ *div.* $CDIM : ECM :: EC^2 - ED^2 : EC^2$,

but $ECM : BMK :: EC^2 : BK^2$,

$\therefore$ *ex æquo* $CDIM : BMK :: EC^2 - EF^2 : BK^2$

$:: FC^2 : BK^2$

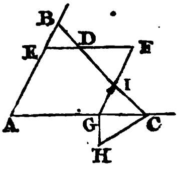
$:: EG^2 : GH^2$, by construction,

$:: EG : GC$,

i. e. in the given ratio.

* * *

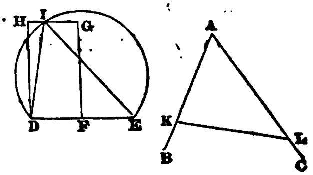
(38.) *Through a given point, between two straight lines containing a given angle, to draw a line which shall cut off a triangle equal to a given figure.*



Let *AB*, *AC* be the lines containing the given angle *BAC*, and *D* the given point. Through *D* draw *DE* parallel to *AC*, and describe a parallelogram *EG* equal to the given figure. Draw *GH* perpendicular to *AC*, and equal to *DE*; and make *HC* =*DF*; join *CD*, and produce it to meet *AB* in *B*; *CB* is the line required.

For the triangles *EBD*, *DIF*, *GIC* being similar, are to one another in the duplicate ratio of the sides *DE*, *DF*, *GC*; but the square of *HC* is equal to the squares of *HG*, *GC*; and $\therefore$ the square of *DF* is equal to the squares of *DE*, *GC*; whence the triangle *DIF* is equal to the triangles DBE, GIC; $\therefore$ the triangle ABC is equal to $AEFG$, *i. e.* to the given figure.

(39.) *Between two lines given in position, to draw a line equal to a given line, so that the triangle thus formed may be equal to a given rectilineal figure.*

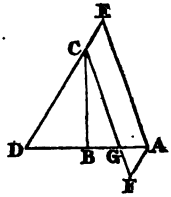


Let AB, AC be the lines given in position, and DE the line whose magnitude is given. Bisect it in F, and on DF describe a rectangular parallelogram equal to the given figure. On DE describe a segment of a circle containing an angle equal to the angle at A, and cutting HG in I. Join DI, IE; and make $AK = ID$, and $AL = IE$. Join KL; it is the line required.

Since $AK = ID$, and $AL = IE$, and the angle at $A = DIE$, $\therefore KL = DE$, and the triangle $AKL = IDE = HGFD =$ the given figure.

(40.) *From two given lines to cut off two others, so that the remainder of one may have to the part cut off from the other a given ratio; and the difference of the squares of the other remainder and part cut off from the first may be equal to a given square.*

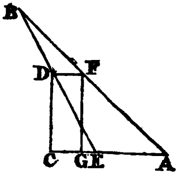
Perpendicular to AB one of the given lines, draw BC equal to a side of the given square; and take AD to the other given line in the given ratio of the part remaining



from the first to the part cut off from the second. Join *DC*; and with the centre *A*, and radius equal to the second given line, describe a circle cutting *DC* in *E*; join *AE*, and draw *CGF* parallel to it, meeting *AF*, drawn parallel to *EC*, in *F*. Then *BG* and *GF* are equal to the parts to be cut off.

For the difference between the squares of *CG*, *GB* is equal to the square of *BC*, *i. e.* to the given square; and $AG : GF :: AD : AE$, *i. e.* in the given ratio.

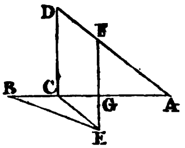
(41.) *From two given lines to cut off two others which shall have a given ratio, so that the difference of the squares of the remainders may be equal to a given square.*



Let *AC* be one of the two given lines. From *C* draw *CD* perpendicular to *AC*, and equal to a side of the given square. Take *AE* to the other given line in the given ratio of the parts to be cut off. Join *ED*, and produce it; and with the centre *A*, and radius equal to that other given line, describe a circle cutting *ED* in *B*. Join *AB*; and let it meet *DF*, which is parallel to *AC*, in *F*. Draw *FG* parallel to *CD*. *CG* and *BF* are the parts required to be cut off.

For $(DF =)\, CG : FB :: EA : AB$, *i. e.* in the given ratio of the parts to be cut off, and the difference between the squares of *FA* and *AG* is equal to the square of *GF*, *i. e.* to the square of *CD*, or the given difference of the squares of the remainders.

(42.) *From two given lines to cut off two others so that the remainders may have a given ratio, and the sum of the squares of the parts cut off may be equal to the square of a given line.*



Let *AB* be one of the given lines, and in it take *AC* to the other given line, in the given ratio of the remainders. From *C* draw *CD* perpendicular to *AB*, and equal to the second given line. Join *AD*, and draw *CE* parallel to *AD*; and with the centre *B*, and radius equal to the side of the given square, describe a circle, cutting *CE* in *E*. Draw *EF* parallel to *DC*. Then *BG*, *GE* will be equal to the parts to be cut off.

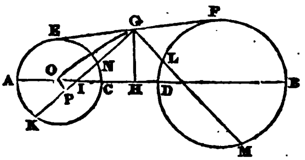
Join *BE*. The squares of *BG*, *GE* are equal to the square of *BE*, *i. e.* to the given square;

and *AG* : *GF* :: *AC* : *CD*,

i. e. in the given ratio of the remainders.

* * *

(43.) *Two points being given in a given straight line; to determine a third such that the rectangles contained by its distances from each extremity and the given point adjacent to that extremity may be equal.*



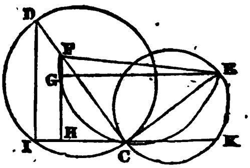
Let *AB* be the given straight line, *C* and *D* the given points in it. On *AC* and *DB* as diameters let circles be described, and let *EF* touch them in *E* and *F*. Bisect *EF* in *G*, and let fall the perpendicular *GH*; *H* is the point required.

From *G* draw any lines *GNK*, *GLM* cutting the circles. Take *O* the centre of the circle *ACE*, and draw

OP perpendicular to GK. (Eucl. ii. 6.) the rectangle NG, GK together with the square of PN is equal to the square of PG; to each of these add the square of PO; and the rectangle NG, GK together with the squares of OP, PN (*i. e.* the square of OC) is equal to the squares of OP, PG, *i. e.* to the square of OG, or to the squares of OH, HG. But the square of OH is equal to the rectangle CH, HA together with the square of OC; whence the rectangle NG, GK is equal to the rectangle CH, HA together with the square of HG. In the same manner it may be shewn that the rectangle LG, GM is equal to the rectangle DH, HB together with the square of HG. But since the rectangle NG, GK, is equal to the rectangle LG, GM, the rectangle CH, HA is equal to the rectangle DH, HB.

Cor. If IH be a mean proportional between CH and HA; $IG = GE$.

(44.) *Through the point of intersection of two given circles, to draw a line in such a manner that the sum of the respective rectangles contained by the parts thereof which are intercepted between the said point and their circumferences, and given lines* A *and* B, *may be equal to a given square.*



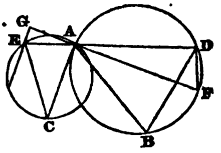
Let the two circles CID, CEK cut each other in the point C; from C draw the diameters CD, CE. In CD take the point F such, that $CD : CF :: A : B$. Join EF; and on it as a diameter describe a semicircle, in which place EG a third proportional to A and the side of the given square. Draw ICK parallel to EG; it will be the line required.

Join FG, and produce it to H. The angle DIC is equal to FGE, *i. e.* to FHC, $\therefore FH$ is parallel to DI;

$$\text{and } CI : CH :: CD : CF :: A : B,$$

$\therefore$ the rectangle A, CH, is equal to the rectangle B, CI. Now since EG is a third proportional to A and the side of the given square, the rectangle A, EG will be equal to the given square. But the rectangle A, EG, is equal to the rectangles A, HC, and A, CK, *i. e.* to the rectangles B, IC, and A, CK; $\therefore$ the rectangles A, KC, and B, IC, are equal to the given square.

(45.) *Through a given point, to draw an indefinite line, such, that if lines be drawn from two other given points, and forming given angles with it, the rectangle contained by the segments intercepted between the given point and the two lines so drawn, shall be equal to the square of a given line.*



Let A be the given point through which the line is to be drawn; B and C the other given points. Join AB, AC; and on them describe segments of circles ADB, AEC, containing angles equal to the given angles. Draw either diameter AF, on which produced take AG such, that the rectangle FA, AG, may be equal to the given square. Draw GE perpendicular to GF; join EA, and produce it both ways; it is the line required.

Join DF. The angles at G and D being right angles, the triangles AGE, ADF are similar,

$$\therefore EA : AG :: FA : AD,$$

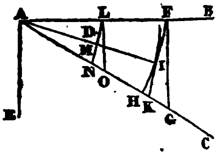
$\therefore$ the rectangle EA, AD is equal to the rectangle FA,

AG, *i. e.* to the given square; and CE, BD form with ED angles equal to the given angles.

If GE does not meet the circle, the problem is impossible.

* * *

(46.) *Through a given point between two straight lines containing a given angle, to draw a line such that a perpendicular upon it from the given angle may have a given ratio to a line drawn from one extremity of it, parallel to a line given in position.*



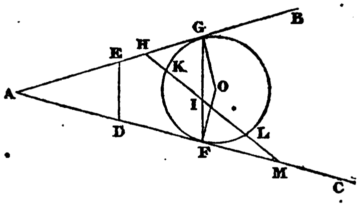
Let AB, AC be the lines forming the given angle BAC, and D a point between them, and AE the line given in position. Draw any line FG parallel to AE, and take $AH : FG$ in the given ratio; and with the centre A, and radius AH, describe a circle, to which draw FIK a tangent. Join AI; and through D draw LMN parallel to FK, and LO parallel to FG. LN is the line required.

For AI is perpendicular to FK, and $\therefore$ AM to LN; and LO is parallel to AE,

$$\text{and } FG : LO :: AF : AL :: (AI =)\ AH : AM,$$
$$\therefore AM : LO :: AH : FG, \textit{ i. e. } \text{in the given ratio.}$$

* * *

(47.) *Through a given point between two indefinite straight lines not parallel to one another, to draw a line which shall be terminated by them, so that the rectangle contained by its segments shall be less than the rectangle contained by the segments of any other line drawn through the same point.*

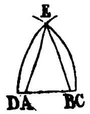


Let *AB*, *AC* be the given lines meeting in *A*. In *AC* take any point *D*, and make *AE*=*AD*. Join *DE*; and through *I* the given point draw *FIG* parallel to *DE*. *FIG* is the line required.

Draw the perpendiculars *FO*, *GO* meeting in *O*. Then since *ED* is parallel to *FG*, and the angles *AED*, *ADE* are equal, ∴ *AFG* and *AGF* are equal. But *AFO*=*AGO*, each being a right angle, ∴ *OGF*=*OFG*, and *OF*=*OG*; a circle ∴ described from the centre *O*, and radius *OG*, will pass through *F*, and touch *AB*, *AC* in *G* and *F*, since the angles at *G* and *F* are right angles. Let now any other line *HKLM* be drawn through *I*, and terminated by *AB*, *AC*. Since all other points in *AB* but *G* are without the circle, *H* is without the circle; ∴ *HM* cuts the circle in *K*; and for the same reason also in *L*. Now the rectangle *KI*, *IL* is equal to the rectangle *GI*, *IF*. But the rectangle *KI*, *IL* is less than the rectangle *HI*, *IM*; ∴ the rectangle *GI*, *IF* is less than the rectangle *HI*, *IM*. In the same manner it may be shewn that the rectangle *GI*, *IF* is less than the rectangle contained by the segments of any other line drawn through *I*, and terminated by *AB*, *AC*.

SECT. VI.

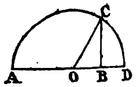
(1.) *To describe an isosceles triangle on a given finite straight line.*



Let AB be the given straight line. Produce it, if necessary; and make AC and BD, each equal to one of the equal sides of the triangle. With A and B as centres, and radii AC, BD, describe circles, cutting each other in E; join AE, BE; AEB is the triangle required.

For $AE = AC = BD = BE$.

(2.) *To describe a square which shall be equal to the difference of two squares, whose sides are given.*

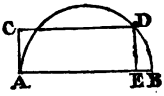


Take a straight line AB terminated at A, and cut off AO equal to a side of the greater, and OB equal to a side of the lesser square. With O as centre, and radius OA, describe a circle OCD; and from B draw BC at right angles to AD. The square described upon BC is the square required.

Join OC. (Eucl. i. 48.), the square described upon BC is equal to the difference of the squares on OC and OB, *i. e.* on AO and OB.

COR. Hence a mean proportional between the sum and difference of two given lines may be determined.

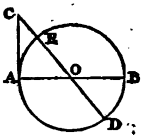
(3.) *To describe a rectangular parallelogram which shall be equal to a given square, and have its adjacent sides together equal to a given line.*



Let *AB* be equal to the given line. Upon it describe a semicircle *ADB*. From *A* draw *AC* perpendicular to *AB*, and equal to a side of the given square. Through *C* draw *CD* parallel to *AB*, and let fall the perpendicular *DE*. The rectangle contained by *AE*, *EB* will be the rectangle required.

For the rectangle *AE*, *EB* is equal to the square of *ED*, which is equal to the square of *AC*, *i. e.* to the given square; and *AB* is the sum of the adjacent sides *AE*, *EB*.

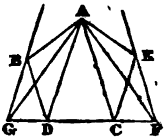
(4.) *To describe a rectangular parallelogram which shall be equal to a given square, and have the difference of its adjacent sides equal to a given line.*



Let *AB* be equal to the given line. On it as diameter describe a circle. From *A* draw *AC* at right angles to *AB*, and ∴ a tangent to the circle at *A*; make *AC* equal to a side of the given square. Take *O* the centre; join *CO*, and produce it to *D*. The rectangle contained by *EC*, *CD* is the rectangle required.

For the rectangle *EC*, *CD*, is equal to the square of *AC*, *i. e.* to the given square; and the difference of the sides containing the rectangle is *ED*=*AB*=the given line.

(5.) *To describe a triangle which shall be equal to a given equilateral and equiangular pentagon, and of the same altitude.*

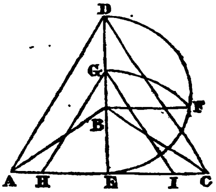


Let *ABDCE* be the given pentagon. Join *AC*, *AD*; and produce *CD* indefinitely both ways. Through *B* and *E* draw *BG*, *EF* respectively parallel to *AD* and *AC*. Join *AF*, *AG*. *AFG* is the triangle required.

Since *AD* is parallel to *BG*, (Eucl. i. 37.) the triangles *ABD*, *AGD* are equal; and for a similar reason, *AEC*=*AFC*; ∴ the triangles *ABD*, *AEC* are equal to *AGD*, *AFC*; to these equals add the triangle *ADC*; and the pentagon *ABDCE* is equal to the triangle *AGF*; and they have the same altitude, *viz*. the perpendicular from *A* upon *DC*.

* * *

(6.) *To describe an equilateral triangle equal to a given isosceles triangle.*



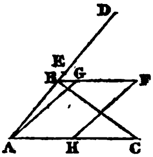
Let *ABC* be the given isosceles triangle. On *AC* describe an equilateral triangle *ADC*, and from *D* draw *DE* perpendicular to *AC*; it will also bisect *AC* and pass through *B*. On *DE* describe a semicircle; and from *B* draw *BF* perpendicular to *DE*, meeting the circle in *F*. With the centre *E*, and radius *EF*, describe a circle meeting *ED* in *G*; draw *GH*, *GI* parallel to *DA*, *DC* respectively; the triangle *GHI* is equilateral, and equal to *ABC*.

Since GH is parallel to AD, and GI to DC, the triangles GHI, ADC are similar; but ADC is equilateral, and ∴ also GHI is equilateral.

Also (Eucl. vi. 8. Cor.) $ED : EG :: EG : EB$,
and (Eucl. vi. 2.) $ED : EG :: EA : EH$,
∴ $EG : EB :: EA : EH$,

and ∴ (Eucl. vi. 15.) the triangles EGH, EBA are equal. But $GHE = GIE$, and $BAE = BCE$, ∴ also $GHI = BAC$.

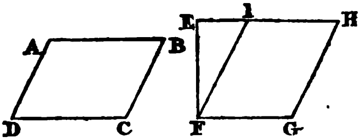
(7.) *To describe a parallelogram, the area and perimeter of which shall be respectively equal to the area and perimeter of a given triangle.*



Let ABC be the given triangle. Produce AB to D, making $BD = BC$; bisect AD in E; draw BF parallel to AC; and with the centre A, and radius AE, describe a circle cutting BF in G. Join AG; and bisect AC in H. Draw HF parallel to AG. $AGFH$ is the parallelogram required.

For $HF = AG = AE$, ∴ HF and AG together are equal to AD, *i.e.* to AB and BC together; and $GF = AH = HC$, ∴ the perimeter of $AGFH$ is equal to the perimeter of ABC; and $AGFH$ is double of a triangle on the base AH and between the same parallels, and ∴ is equal to the triangle ABC.

(8.) *To describe a parallelogram which shall be of given altitude, and equiangular and equal to a given parallelogram.*

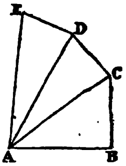


Let *ABCD* be the given parallelogram, and *EF* the given altitude. Draw *EH* and *FG* at right angles to *FE*; and at the point *F*, in the line *GF*, make the angle *GFI* equal to *CDA*; take *FG* a fourth proportional to *FI*, *AD* and *DC*; and from *G* draw *GH* parallel to *FI*, meeting *EH* produced in *H*; *IFGH* is the parallelogram required.

For its altitude is *EF*; and the angle *GFI*=*CDA*, ∴ *FIH*=*DAB*; whence the parallelograms are equiangular; and they are equal; since the sides about the equal angles are reciprocally proportional (Eucl. vi. 14.).

* * *

(9.) *To describe a square which shall be equal to the sum of any number of given squares.*



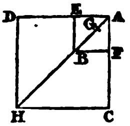
Let *AB* be a side of one of the given squares. From *B* draw *BC* perpendicular to *AB*, and equal to a side of the second square. Join *AC*; and from *C* draw *CD* perpendicular to it, and equal to a side of the third square. Join *AD*; and from *D* draw *DE* perpendicular to *AD*, and equal to a side of the fourth. Join *AE*. The square of *AE* is equal to the squares of *AB*, *BC*, *CD*, *DE*.

Since the angles *ADE*, *ACD*, *ABC* are right angles, the square of *AE* is equal to the squares of *AD*, *DE*, *i. e.* to the squares of *AC*, *CD*, *DE*; and ∴ to the squares of *AB*, *BC*, *CD*, *DE*.

And by proceeding in the same manner whatever be the number of given squares, one equal to their sum may be found.

Cor. Hence lines may be found, which have the same ratio as the square roots of the natural numbers.

(10.) *Having given the difference between the diameter and side of a square; to describe the square.*

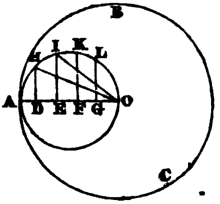


Let AB be the given difference. Draw AC, AD, each making half a right angle with AB; and complete the square EF. Take $AG =$ the difference between BA and BF. Since the ratio between the side of a square and its diameter is given, that of their difference to the diameter is also given. Take $\therefore AH : AB :: AB : AG$; and through H draw HC, HD perpendicular to AC, AD; CD is the square required.

For DC being a parallelogram is also (Eucl. i. 46. Cor.) rectangular; and the angle DAH being half a right angle, is equal to DHA, $\therefore DA = DH$; whence the sides are equal; and the figure is a square. And since $BG = BF$, $HB = HC$; and AB is the difference between the diameter and side.

(11.) *To divide a circle into any number of concentric equal annuli.*

Let ABC be the given circle, and O its centre. Draw any radius OA, and divide into the given number



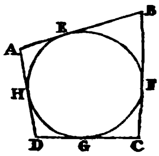
of equal portions in the points *D, E, F, G*, &c. On *OA* describe a semicircle, and draw the perpendiculars *DH, EI, FK, GL*, &c. Join *OH, OI*, &c. and with the centre *O* and radii *OH, OI*, &c. let circles be described; they will divide the circle *ABC* as is required.

Since the areas of circles are in the duplicate ratio of their radii; the area of the circle whose radius is *OA* is to that whose radius is *OH* in the duplicate ratio of $OA : OH$, *i. e.* in the ratio of $OA : OD$; $\therefore$ the area of the first annulus will be to the area of the circle whose radius is $OD :: AD : OD$. And in the same manner the area of the second annulus, will be to the area of the circle whose radius is *OD*, as $DE : OD$; and since $AD = DE$, the annuli will be equal. The same may be proved of all the rest.

Cor. The construction will be nearly the same, if it be required to divide the circle into annuli which shall have a given ratio; by dividing the radius *AO* in that proportion.

* * *

(12.) *In any quadrilateral figure circumscribing a circle, the opposite sides are equal to half the perimeter.*

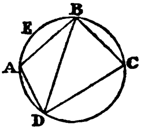


Let *ABCD* be a quadrilateral figure circumscribing the circle *EFG*; its opposite sides are equal to half the perimeter.

For (Eucl. iii. 36. Cor.) $AE = AH$, and $DH = DG$, $\therefore$ *AD* is equal to *AE* and *DG* together. In the same way *BC* is equal to *BE* and *GC* together, $\therefore$ *AD* and *BC* together are equal to *AB* and *DC* together.

* * *

(13.) *If the opposite angles of a quadrilateral figure be equal to two right angles, a circle may be described about it.*

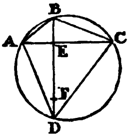


Let *ABCD* be a quadrilateral figure, whose opposite angles are equal to two right angles.

Join *BD*; then if a circle be described about the triangle *BCD* it will pass through *A*. For the angle *BCD* and the angle in the segment *BED*, are together equal to two right angles, and ∴ equal to *BCD*, *BAD*; whence *BAD* is equal to the angle in the segment *BED*; and ∴ *A* must be a point in the circumference; or the circle will be described about *ABCD*.

* * *

(14.) *A quadrilateral figure may have a circle described about it, if the rectangles contained by the segments of the diagonals be equal.*

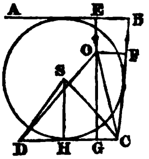


Let *ABCD* be a quadrilateral figure, the rectangles contained by the segments of whose diagonals are equal, *viz.* the rectangle *AE*, *EC*, equal to *BE*, *ED*.

Describe a circle about the triangle *ABC*; if it does not pass through *D*, let it cut *BD* in *F*; then (Eucl. iii. 35.) the rectangle *BE*, *EF*, is equal to the rectangle *AE*, *EC*, *i. e.* to the rectangle *BE*, *ED*, by the supposition; whence *EF* is equal to *ED*, the less to the greater, which is impossible; ∴ the circle must pass through *D*.

* * *

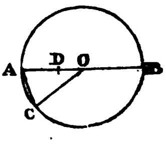
(15.) *If from any point within a regular figure circumscribed about a circle perpendiculars be drawn to the sides; they will together be equal to that multiple of the semidiameter, which is expressed by the number of the sides of the figure.*



Let *ABCD* be a regular figure circumscribed about the circle; and from any point *O*, let perpendiculars *OE*, *OF*, *OG*, &c. be drawn. Take *S* the centre of the circle. Join *SD*, *SC*, *SH*. Then the figure will be divided into as many triangles round *S* and *O*, as there are sides of the figure; now the triangle *SCD* : *OCD* :: *SH* : *OG*; and the same being true of the triangles on each side, the sum of the triangles round *S*, will be to the sum of the triangles round *O*, as the sum of the lines *SH* to the sum of the perpendiculars from *O*. And the first term of the proportion being equal to the second, the sum of the perpendiculars from *O* is equal to that multiple of the radius which is expressed by the number of the sides; each perpendicular from *S* being a radius of the circle.

* * *

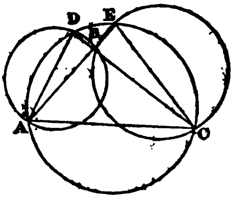
(16.) *If the radius of a circle be cut in extreme and mean ratio; the greater segment will be equal to the side of an equilateral and equiangular decagon inscribed in that circle.*



Let *AO*, the radius of the circle *ABC*, be cut in extreme and mean ratio in *D*; *AD* is equal to the side of an equilateral and equiangular decagon inscribed in the circle. In the circle place *AC* equal to *AD*; join *CO*. Then (Eucl. iv. 10.) the angles at *A* and *C* are double the angle at *O*; whence *AOC* is one fifth part of two right angles, or one tenth part of four right angles, *i. e.* of the angles at *O*; and ∴ *AC* is the side of a regular decagon inscribed in the circle.

◆

(17.) *Any segment of a circle being described on the base of a triangle; to describe on the other sides segments similar to that on the base.*



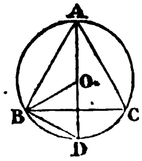
Let *ABC* be a triangle, on the base *AC* of which a segment of a circle *ADC* is described. Produce *AB*, *CB* to *E* and *D*. Join *AD*, *CE*; and through *A*, *D*, *B*, and *C*, *E*, *B* let circles be described; the segments *ADB*, *BEC* are similar to *ADC*.

For the angle *ADC* being in the segments *ADB*, *ADC*, those segments are similar. For the same reason the segments *ADC*, *BEC* are similar. And since the angles *ADC*, *AEC* are equal, ∴ the segments *ADB* *BEC* are similar.

◆

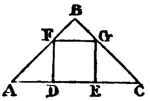
(18.) *If an equilateral triangle be inscribed in a circle; the square described on a side thereof is equal to three times the square described upon the radius.*

Let *ABC* be an equilateral triangle inscribed in a circle. From *A* draw the diameter *AD*, and take *O* the centre;



join BD, BO. Then the angle $BOD = BAC = BCA = BDO$, $\therefore BD = BO$; and the squares of AB, BD are equal to the square of AD, *i. e.* to four times the square of BO, or BD; and $\therefore$ the square of AB is equal to three times the square of BD or BO.

(19.) *To inscribe a square in a given right-angled isosceles triangle.*

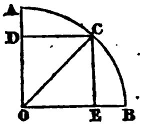


Let ABC be a right-angled isosceles triangle, having the side $BA = BC$. Trisect the hypothenuse AC in the points D, E; and from D, E draw DF, EG perpendicular to AC; join FG; $DFGE$ is the square required.

Since the angle DAF is half a right angle, and the angle at D a right angle, $\therefore$ the angle DFA is half a right angle, and equal to DAF; whence $DF = DA$. In the same manner it may be shewn that $EG = EC$. But $AD = EC$; and $\therefore FD$, DE and EG are equal; and (Eucl. i. 33.) $FG = DE$; $\therefore$ the figure is equilateral. And it is rectangular, (Eucl. i. 46.) since the angles at D and E are right angles; $\therefore$ it is a square.

(20.) *To inscribe a square in a given quadrant of a circle.*

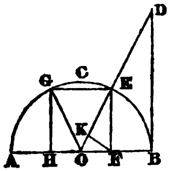
Let AOB be the given quadrant, whose centre is O. Bisect the angle AOB by the line OC. Draw CE, CD parallel to OA, OB. DE is a square.



For the angle $COD=COE$, and $CDO=CEO$, since each of them with DOE make angles equal to two right angles, and CO is common, ∴ $CE=CD$. And by construction $CE=OD$, and $OE=CD$, ∴ the figure is equilateral. And the angle DOE is a right angle, ∴ (Eucl. i. 46. Cor.) all its angles are right angles; and consequently the figure is a square.

* * *

(21.) *To inscribe a square in a given semicircle.*



Let ACB be the given semicircle; take O its centre, and from B draw BD perpendicular and equal to BA. Join OD, cutting the circumference in E; and from E draw EF perpendicular to AB, and EG parallel to it; draw GH parallel to EF. Then EH is the square required.

Join OG. Since EG is parallel to AB, the angle $GOH=EOF$, and the angles at H and F are right angles, and $GO=OE$, ∴ $HO=OF$.

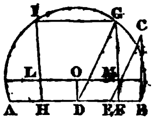
Now $EF : FO :: DB : BO$,

∴ $EF=2FO=FH$; the figure is ∴ equilateral; and it is, by construction, rectangular; ∴ it is a square.

Cor. Since $FE=2FO$, $FE^2=4OF^2$, and $OE^2=5OF^2$; and if FK be drawn perpendicular to OE, $OE : OK :: 5 : 1$.

* * *

(22.) *To inscribe a square in a given segment of a circle.*



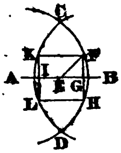
Let AIB be the segment of a circle, whose base AB is bisected in D. From B draw BC perpendicular to BA and equal to BD. Bisect BD in E, and join CE. Draw DG parallel to CE, and GF to CB. Take $DH = DF$; draw HI perpendicular to AH, and ∴ parallel to GF; Join GI. FI is the square required.

Since GD and GF are respectively parallel to CE and CB,

$$GF : FD :: CB : BE :: 2 : 1,$$

∴ $GF = 2FD = FH$. Take O the centre, and draw OL, OM perpendiculars to HI, FG; then since $HD = DF$, $OL = OM$, ∴ (Eucl. iii. 14.), $IL = GM$; but $LH = FM$, ∴ $IH = GF$; whence $IG = HF$, and the figure is equilateral; and since the angle at F is a right angle, the figure is rectangular, and ∴ is a square.

(23.) *Having given the distance of the centres of two equal circles which cut each other; to inscribe a square in the space included between the two circumferences.*

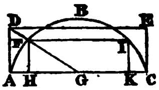


Let A and B be the centres of two equal circles, which cut each other in C and D. Join AB, and bisect it in E; and at the point E make the angle GEF = half a right angle; and from F draw FGH perpendicular to AB. Make $EI = EG$; and through I draw KL perpendicular to AB; join KF, LH. KH is a square.

Since $EI = EG$, $BI = AG$, and ∴ (Eucl. iii. 14.) $KL = FH$; and they are parallel, ∴ KF is equal and parallel to LH, ∴ KH is a parallelogram. Also since GEF is half a right angle, and EGF a right angle, ∴ EFG is half a right angle, and ∴ equal to GEF; whence $EG = GF$, and $FH = IG$. But KF is equal and parallel to IG (Eucl. i. 33); ∴ the four sides are equal; and GFK is a right angle, ∴ the figure is rectangular (Eucl. i. 46. Cor.), and consequently is a square.

* * *

(24.) *In a given segment of a circle to inscribe a rectangular parallelogram, whose sides shall have a given ratio.*



Let ABC be the given segment of a circle. From A draw AD perpendicular to AC, and make $AD : AC$ in the ratio of the sides. Complete the parallelogram AE. Bisect AC in G, and join DG; and from F draw FH perpendicular, and FI parallel to AC. Draw IK parallel to FH; HI is the rectangular parallelogram required.

Since FH is perpendicular to AC, it is parallel to AD;

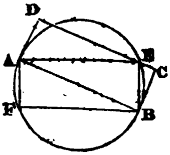
and ∴ $FH : HG :: AD : AG$,

whence $FH : HK :: AD : AC$,

i. e. in the given ratio. And FHG being a right angle all the angles of the figure are right angles.

* * *

(25.) *In a given circle to inscribe a rectangular parallelogram equal to a given rectilineal figure.*



Let AEB be the given circle; on the diameter AB describe a rectangular parallelogram $ABCD$ equal to the given rectilineal figure; and let the side DC

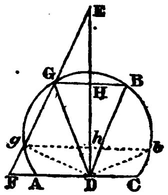
cut the circumference in *E*. Join *AE, EB*. Draw *BF* parallel to *AE*, and join *AF*. *FE* is the rectangular parallelogram required.

For the angle *EBF* is equal to the two angles *EBA*, *ABF*, *i. e.* to *EBA*, *BAE*, and ∴ is a right angle; and the angles *BEA* and *BFA* are right angles, by construction; ∴ also *EAF* is a right angle; the figure *AFBE* is therefore rectangular; and it is double of *ABE*, and (Eucl. i. 41.) ∴ equal to *ABCD*, *i. e.* to the given rectilineal figure.

The given figure must not exceed the square of half the diameter.

* * *

(26.) *In a given segment of a circle to inscribe an isosceles triangle, such that its vertex may be in the middle of the chord, and the base and perpendicular together equal to a given line.*



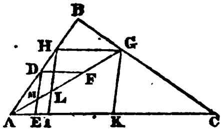
Let *ABC* be the given segment. Bisect *AC* in *D*, and draw *DE* at right angles to *AC*, and equal to the given line. Make *DF* the half of *DE*; and join *EF*, meeting the circle in *G*. Draw *GB* parallel to *AC*; join *GD*, *DB*. *GDB* is the triangle required.

Since *GB* is parallel to *AC*, it is bisected by *DE*. Also (Eucl. vi. 2.) *EH* is double of *GH*, and ∴ equal to *GB*; ∴ *GB* and *HD* together are equal to *ED*, *i. e.* to the given line; and since *GH*=*HB*, and the angles at *H* are right angles, *GD*=*DB*, ∴ the triangle is isosceles.

If *EF* does not meet the segment, the problem is impossible. When the line FE cuts the segment, there are two isosceles triangles DGB, Dgb that will answer the conditions; when it touches the segment, only one.

* * *

(27.) *In a given triangle to inscribe a parallelogram similar to a given parallelogram.*



Let ABC be the given triangle. In AB take any point D, and draw DF parallel to AC; and make the angle FDE equal to one angle of the parallelogram, and take $DF : DE$ in the ratio of the sides. Join AF, and produce it to G; draw GH, HI respectively parallel to FD, DE; and GK parallel to HI. HK is the parallelogram required.

For HI being parallel to DE, and HG to DF,

$$HI : DE :: HA : DA :: HG : DF,$$

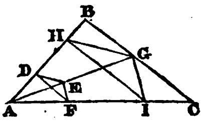
$$\therefore HI : HG :: DE : DF,$$

i. e. in the ratio of the sides.

Also the angle $GHI = FDE =$ one of the angles of the parallelogram, $\therefore$ HIK will be equal to the adjacent angle of the parallelogram, and HK is similar to the given parallelogram.

* * *

(28.) *In a given triangle to inscribe a triangle similar to a given triangle.*



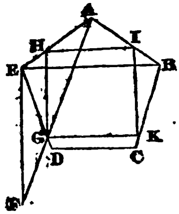
Let ABC be the given triangle, in which the triangle is to be inscribed. In AB take any point D, and draw any line DF to the opposite side; and at the points *D* and *F* make the angles *FDE*, *DFE* equal to two of the angles of the given triangle to which the inscribed one is to be similar, ∴ the angle at *E* will be equal to the third angle. Join *AE*, and produce it to *G*; and from *G* draw *GH*, *GI* respectively parallel to *ED*, *EF*; join *HI*. *HIG* is the triangle required.

Since *DE* and *EF* are respectively parallel to *HG*, *GI*, the angle *DEF* is equal to *HGI*.

Also $DE : HG :: AE : AG :: EF : GI$, whence (Eucl. vi. 6.) the triangles *HGI*, *DEF* are similar, and ∴ *HGI* is similar to the given triangle.

* * *

(29.) *In a given equilateral and equiangular pentagon, to inscribe a square.*



Let *ABCDE* be the given pentagon. Join *EB*; and from *E* draw *EF* perpendicular and equal to *EB*. Join *AF*; and from *G*, where it cuts *ED*, draw *GH* parallel to *FE*. Draw *HI*, *GK* parallel to *EB*. Join *IK*. *HK* is the square required.

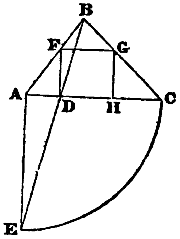
Since *HG* is parallel to *EF* and *HI* to *EB*,

$$HG : EF :: AH : AE :: HI : EB,$$

but $EF = EB$, ∴ $HG = HI$. And since $AE = AB$, ∴ (Eucl. vi. 2.) $HE = IB$; also *GK* and *DC* being parallel to *EB*, and $DE = BC$, ∴ $EG = BK$. The triangles *EHG*, *IKB*, therefore have two sides in each and the included angles equal, and ∴ $HG = IK$, and the angle $EHG = BIK$, whence *HG* and *IK* are also parallel; therefore also *GK* is equal to *HI*; hence the four sides are equal; and the angle at *H* being a right angle, all the angles are right angles, and consequently *HK* is a square.

* * *

(30.) *In a given triangle to inscribe a rhombus, one of whose angles shall be in a given point in the side of the triangle.*



Let ABC be the given triangle, and D the given point. Join BD, and produce it; and with the centre A, and radius AC, describe a circle cutting it in E. Join AE; and draw DF parallel to it, FG parallel to AC, and GH to FD. FH is the rhombus required.

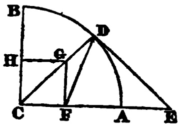
Since FD is parallel to AE, $BF : FD :: BA : AE$; and since FG is parallel to AC,

$$BF : FG :: BA : AC :: BA : AE,$$

$\therefore FD = FG$; and the sides opposite to these are equal, $\therefore$ the figure $FDHG$ is a rhombus.

* * *

(31.) *To inscribe a circle in a given quadrant.*



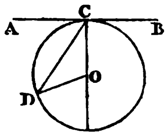
Let ABC be the given quadrant. Bisect the angle ACB by the line CD; and at D draw DE touching the quadrant, and meeting CA produced in E. Make $CF = AE$. From F draw FG at right angles to AC. G is the centre of the circle required.

From G draw GH perpendicular to BC. Join DF. Since the angle DCE is half a right angle, and the angle at D a right angle, $DE = DC = AC = FE$, $\therefore$ the angle $EDF = EFD$; whence also $GDF = GFD$, and $GD = GF$; and since the angles FCG, GCH are equal, and GC common to the right-angled triangles GFC, GHC, $\therefore GF = GH$; $\therefore$ the three lines GD, GF, GH

are equal and the circle described from the centre *G*, and distance of any one of them, will pass through the extremities of the other two, and touch the arc and sides in the points *D*, *F*, *H*, because the angle at those points are right angles.

* * *

(32.) *To describe a circle, the circumference of which shall pass through a given point, and touch a given straight line in a given point.*



Let *AB* be the given straight line, *C* the given point, in which the circle is to touch it, *D* the point through which it must pass. Draw *CO* perpendicular to *AB*. Join *CD*; and at the point *D* make the angle *CDO*=*DCO*; the intersection of the lines *CO* and *DO* is the centre of the circle required.

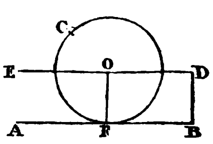
Since the angle *DCO*=*CDO*, *CO*=*DO*, and ∴ a circle described from the centre *O*, at the distance *OD*, will pass through *C*, and touch the line *AB* in *C*, bebecause *OC* is perpendicular to *AB*.

* * *

(33.) *To describe a circle which shall pass through a given point, have a given radius, and touch a given straight line.*

Let *AB* be the given straight line, and *C* the given point through which the circle must pass.

In *AB* take any point *B*; and from

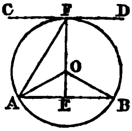


it draw *BD* at right angles to *AB*, and equal to the given radius; through *D* draw *DE* parallel to *AB*; and with the centre *C*, and radius equal to the given radius, describe a circle cutting *DE* in *O*. *O* is the centre of the circle required.

From *O* draw *OF* perpendicular to *AB*, it is equal to *DB*, *i. e.* to the given radius; and the circle described from the centre *O*, and radius *OF*, will touch (Eucl. iii. 16. Cor.) the line *AB* in *F*, and pass through *C*.

* * *

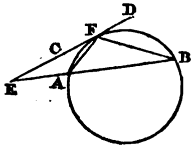
(34.) *To describe a circle which shall pass through two given points, and touch a given straight line.*



Let *A*, *B* be the given points, and *CD* the given straight line. Join *AB*. And 1. let *CD* be parallel to *AB*.

Bisect *AB* in *E*, and draw *EF* perpendicular to *AB*, and ∴ to *CD*. Join *FA*, and make the angle *FAO*=*AFO*; then will *O* be the centre of the circle required.

Since the angle *FAO*=*AFO*, *AO*=*OF*. But *AE*=*EB*, and *EO* is common to the triangles *AEO*, *BEO*, and the angles at *E* right angles, ∴ *AO*=*OB*. Whence *AO*, *OB*, *OF* are all equal; and the circle described from the centre *O*, at the distance of any one of them, will pass through the extremities of the other two, and touch the line *CD*, since *OF* is perpendicular to *CD*.

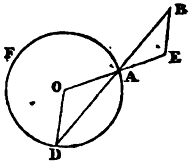


2. But if *AB* is not parallel to *CD*, let them be produced to meet in *E*; and take *EF* a mean proportional between *EA* and *EB*. Join *FA*, *FB*; and describe a circle about the triangle *AFB*; it will be the circle required.

Since *EF* is a mean proportional between *EA* and *EB*, *EF* touches the circle (Eucl. iii. 37.), which passes through *A* and *B*.

* * *

(35.) *To describe a circle, the circumference of which shall pass through a given point, and touch a circle in a given point; the two points not being in a tangent to the given circle.* Chronicle 961



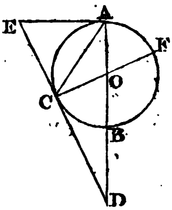
Let *A* be the given point in the circumference of the circle whose centre is *O*; *B* the given point without. Join *BA*, and produce it to *D*. Join *OD*; and through *A* draw *OAE*; and draw *BE* parallel to *OD*, cutting *OAE* in *E*. *E* is the centre of the circle required.

Since (Eucl. i. 29.) the angle *ODA* is equal to *ABE*, and *OAD* to *BAE*, ∴ the triangles *ODA*, *ABE* are similar, and *OD* being equal to *OA*, *AE* will be equal to *EB*; a circle ∴ described with the centre *E*, and radius *EA*, will pass through *B*, and touch the circle *ADF* in the point *A*, since the line joining the centres passes through *A*.

* * *

(36.) *To describe a circle the centre of which may be in the perpendicular of a given right-angled triangle, and the circumference pass through the right angle and touch the hypothenuse.*

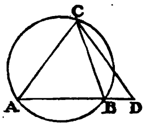
Let *EAD* be the given right-angled triangle, having the angle at *A* a right angle. Make *EC* = *EA*. Join



CA; and draw CO at right angles to ED. The circle described with O as centre, and radius OA, will be the circle required.

Since $EA = EC$, the angle $ECA = EAC$, and ECO, EAO are equal, being right angles; $\therefore$ $OCA = OAC$ and $OA = OC$. The circle $\therefore$ described from the centre O, and radius OA, will pass through the extremity of OC, and touch ED in C, because CO is at right angles to ED.

(37.) *To describe a circle which shall pass through the extremities of a given line, so that if from any point in its circumference a line be drawn making a given angle with the given line; the rectangle contained by the segment it cuts off and the given line, may be equal to the square of the line drawn from the same point to the further extremity of the given line.*



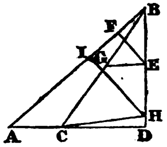
Let AB be the given line. On it describe a segment of a circle containing an angle equal to the given angle. Complete the circle; it will be the one required.

From any point C draw CD, making with AB the angle ADC equal to the given angle; join CA, CB. Since the angle $CDA = ACB$, and the angle at A is common, the triangles ACD, ABC are equiangular, and therefore

$$AB : AC :: AC : AD,$$

whence the rectangle contained by AB, AD, is equal to the square of AC.

(38.) *To determine a point in the perpendicular let fall from the vertical angle of a triangle on the base; about which as a centre a circle may be described touching the longer side, and passing through the opposite angular point.*



Let ABC be a triangle, and from B the vertex let BD be drawn perpendicular to AC. In DB take any point E, and from it draw EF perpendicular to AB; and from E to BC, draw $EG=EF$; from C draw CH parallel to GE, and from H draw HI perpendicular to AB; H is the point required.

Since EF is parallel to HI,

$$FE : HI :: BE : BH,$$

and since GE is parallel to HC,

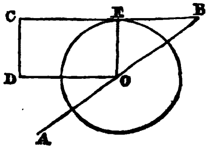
$$GE : HC :: BE : BH,$$

$$\therefore FE : HI :: GE : HC;$$

but, by construction, $FE=EG$, $\therefore$ $HI=HC$; and a circle described from the centre H at the distance HI, will pass through C, and touch AB in I, since the angle HIB is a right angle.

* * *

(39.) *To describe a circle which shall have a given radius, and its centre in a given straight line, and shall also touch another given straight line inclined at a given angle to the former.*



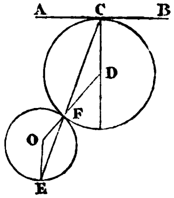
Let AB be the given line, in which the centre is to be; BC the line which the circle is to touch.

In BC take any point C, and draw CD at right angles to it; and make

CD equal to the given radius. Through *D* draw *DO* parallel to *CB*; *O* is the centre of the circle required.

Through *O* draw *OE* parallel to *DC*; ∴ *CO* is a parallelogram; whence *OE* is equal to *DC*, *i. e.* to the given radius. With the centre *O*, and radius *OE*, describe a circle; it will touch *CB* in *E*, because *CO* being a parallelogram, and *ECD* a right angle, *CEO* is also a right angle.

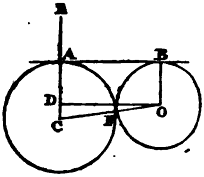
(40.) *To describe a circle, which shall touch a straight line in a given point, and also touch a given circle.* See (35)



Let *AB* be the given line, and *C* the given point in it, *O* the centre of the given circle. Draw *CD* perpendicular to *AB*, and *OE* parallel to *CD*. Join *CE*, meeting the circumference in *F*. Join *OF*, and produce it to meet *CD* in *D*. *D* is the centre of the circle required.

Since the triangles *OEF*, *CFD* are similar, and *OE* $=$ *OF*, ∴ *FD* $=$ *DC*; consequently a circle described with the centre *D*, and radius *DF*, will pass through *C*, and touch *AB* in *C*, because the angles at *C* are right angles; and it will touch the given circle in *F*, since the line joining the centres passes through *F*.

(41.) *To describe two circles, each having a given radius, which shall touch each other, and the same given straight line on the same side of it.*

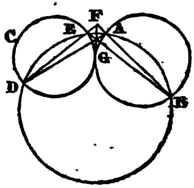


Let *AB* be the given straight line. From any point *A* in it, draw *AC* at right angles to it, and make *AC*, *AD*, equal to the given radii. Produce *CA* to *E*, making $AE = AD$. Draw *DO* parallel to *AB*; and with the centre *C*, and radius *CE*, describe a circle cutting *DO* in *O*. *C* and *O* will be the centres of the circles required.

Join *CO*; and draw *OB* perpendicular to *AB*; then *DAB* being a right angle, as also *ABO*, ∴ *AD* is parallel to *BO*; and *DO* was drawn parallel to *AB*, ∴ *AO* is a parallelogram, and $OB = AD$. With the centres *C* and *O*, and radii *CA*, *OB* describe circles, they will touch *AB*, since the angles at *A* and *B* are right angles; they will also touch each other, for *CO* is equal to *CE*, or to *CA* and *AE*, *i. e.* to *CA* and *AD*, or the sum of the radii.

* * *

(42.) *To describe a circle passing through two given points, and touching a given circle.*

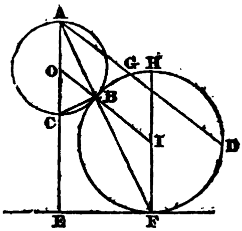


Let *A* and *B* be the given points, and *CDE* the given circle. Describe a circle through *A* and *B*, and cutting the given circle in *D* and *E*. Join *DE*, *EB*, *DA*, *AB*. Then the angle *EDA* $= EBA$; if ∴ *DE* and *BA* be produced to meet in *F*, the triangle *FDA* will be similar to the triangle *FBE*;

$$\text{and} \therefore DF : FA :: BF : FE,$$

or the rectangle *DF*, *FE* is equal to the rectangle *AF*, *FB*. Draw *FG* a tangent to the given circle; then the square of FG is equal to the rectangle EF, FD, and $\therefore$ to the rectangle BF, FA; whence a circle described through the points A, G, B, will touch the given circle, since it touches FG.

(43.) *To describe a circle, which shall pass through a given point, and touch a given circle and a given straight line.*



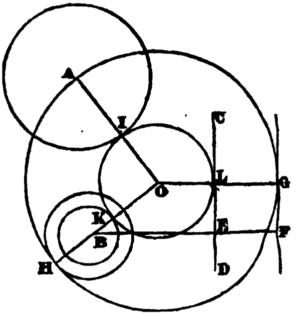
Let ABC be the given circle, D the given point, and EF the given straight line. Through O draw AOE perpendicular to EF. Join AD; and divide it in G, so that the rectangle AG, AD, may be equal to the rectangle AC, AE. Through G and D describe a circle touching EF in F; this will also touch the circle ABC.

Draw the diameter FH; it is (Eucl. iii. 18.) parallel to AE. Join AF, meeting the circle in B. Join CB. The triangles ABC, AEF having the angle at A common, and the angles ABC, AEF right angles, are similar; whence

$$AC : AB :: AF : AE,$$

$\therefore$ the rectangle AB, AF is equal to the rectangle AC, AE, *i. e.* to the rectangle AG, AD; $\therefore$ B is a point in the circle HDF. Take I the centre; join OB, BI. Since AC is parallel to FI, the angle $OAB = BFI$; but $OAB = OBA$, and $IFB = IBF$, $\therefore$ $OBA = IBF$; and OBI is a straight line, which joins the centres of the two circles, which $\therefore$ touch each other.

(44.) *To describe a circle which shall touch a straight line and two circles given in magnitude and position.* Chronicle 963



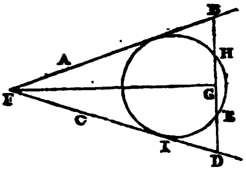
Let *A* and *B* be the centres of the two circles, and *CD* the line given in position. From *B* let fall the perpendicular *BE*, and produce it, making *EF*= the radius of the circle whose centre is *A*. Through *F* draw *FG* parallel to *CD*. With the centre *B*, and radius equal to the difference of the radii of the two circles, describe a circle; through *A* let a circle be described, touching the line *GF* and the last described circle (vi. 43.); and let *G* and *H* be the points of contact. The centre of this circle will also be the centre of the circle required.

Let *O* be the centre; join *OA*, *OG*, *OH*; and with the centre *O*, and radius *OI*, describe the circle *IKL*. Since $LG=KH=AI$, $\therefore OL=OK=OI$; the circle *IKL* $\therefore$ touches *CD* in *L*, and the circle, whose centre is *A*, in *I*; and since *OB* is equal to the difference between *OH* and *HB*, *i. e.* between *OA* and $(IA-BK)$, or is equal to *OK* and *KB* together, $\therefore$ it touches the circle whose centre is *B*, in *K*.

* * *

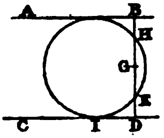
(45.) *To describe a circle which shall touch two given straight lines, and pass through a given point between them.*

Let *AB*, *CD* be the given lines, and *E* the given



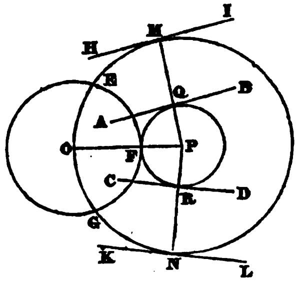
point. Produce the lines to meet in *F*. Bisect the angle *BFD* by the line *FG*; and from *E* draw *EG* perpendicular to *FG*, and produce it both ways to *B* and *D*. Take *GH*=*GE*; and make *DI* a mean proportional between *DE* and *DH*; a circle described through the points *H*, *E*, *I*, will touch *CD*.

For the rectangle *DE*, *DH*, is equal to the square of *DI*. And for a similar reason it will touch *AB*; since the rectangle *BH*, *BE*, is equal to the rectangle *ED*, *DH*.



If the lines *AB*, *CD* be parallel; through the given point *E*, draw *DEHB* perpendicular to *AB* or *CD*; bisect it in *G*, and make *GH*=*GE*. Take *DI* a mean proportional between *DE* and *DH*; and a circle described through *I*, *E* and *H* will be the circle required.

(46.) *To describe a circle which shall touch two given straight lines, and also touch a given circle.* Chronicle

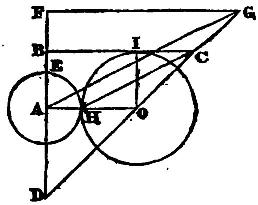


Let *AB*, *CD* be the given straight lines, *EFG* the given circle, whose centre is *O*. Draw *HI*, *KL* parallel to the given lines, so that their perpendicular distances from those lines may be equal to *OF* the radius of the given circle. By the last problem describe a circle touching *HI*, *KL*, and passing through *O* the centre of the given circle. Let *P* be the centre of this circle; it will also be the centre of the circle required.

Join *PM, PN, PO.* Since these lines are equal, and *MQ, RN, OF* are also equal by construction, ∴ *PQ, PR, PF* are also equal; and a circle described from the centre *P* at the distance of any one of them, will pass through the extremities of the other two, and touch the lines *AB, CD,* in *Q* and *R*; since the angles at those points are equal to the angles at *M* and *N,* and ∴ right angles; and it will also touch the circle *EFG* in *F,* since *OP* the line joining the centres passes through *F.*

* * *

(47.) *To describe a circle which shall touch a circle and straight line, both given in position, and have its centre also in a given straight line.*



Let the circle whose centre is *A,* and the straight line *BC* be given in position; and let *CD* be the line, in which the centre of the required circle is to be. On *BC* let fall the perpendicular *AB*; and make *BF* = *AE*; through *F* draw *FG* parallel to *BC,* meeting *DC* in *G.*

Join *GA*; and draw *CH* parallel to it, meeting the given circle in *H,* (if the problem be possible). Join *AH,* and let it meet *DC* in *Q.* *O* is the centre of the circle required.

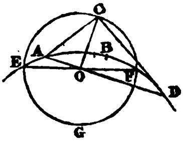
Let fall the perpendicular *OI.* Then (Eucl. vi. 2.)

$HO : OC :: AH : GC :: FB : GC$ by construction,
$:: BD : DC,$ (Eucl. vi. 2.)
$:: IO : OC,$ by sim. triangles;

$\therefore HO = IO$; and a circle described with the centre *O*, and radius *OI* or *OH*, will pass through the extremity of the other, and touch the line *BC* in *I*, and the circle in *H*; because the angles at *I* are right angles; and *AO* the line joining the centres of the circles passes through *H*.

* * *

(48.) *Through two given points within a given circle, to describe a circle, which shall bisect the circumference of the other.*



Let *A* and *B* be the given points within the circle whose centre is *O*. Join *AO*; and produce it indefinitely; and from *O* draw *OC* at right angles to it. Join *AC*; and draw *CD* at right angles to it, meeting *AO* produced in *D*; and through *A*, *B*, *D* describe a circle; it will bisect the other in the points *E*, and *F*.

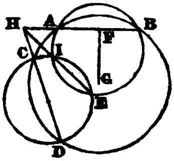
Join *EO*, *OF*. Then (Eucl. vi. 8.)

$$AO : OC :: OC : OD,$$

$\therefore$ the rectangle *AO*, *OD* is equal to the square of *OC*, *i. e.* to the rectangle *EO*, *OF*; whence (Eucl. iii. 35.) *EOF* is a straight line; and since it passes through the centre of the circle *ECF*, it will be a diameter of that circle; $\therefore$ the circumference *ECF* is equal to the circumference *EGF*, or the circumference of the given circle is bisected.

* * *

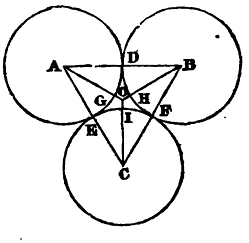
(49.) *Through two given points without a given circle, to describe a circle, which shall cut off from the given one an arc equal to a given arc.*



Let *A* and *B* be the given points, and *CDE* the given circle. Join *AB*; and bisect it in *F*; from *F* draw *FG* at right angles to *AB*; and from any point *G* in it, at the distance *GA* or *GB*, describe a circle *ABD*, cutting the given circle in *C* and *D*. Join *DC*; and produce it to meet *BA* in *H*. From *H* draw *HIE* (ii. 20.), so that *IE* may be equal to the chord of the given arc. Through *A*, *B* and *E* describe a circle; it will also pass through *I*, and cut off the arc required.

For the rectangle *HI*, *HE* is equal to the rectangle *HC*, *HD*, and ∴ also to the rectangle *HA*, *HB*, whence *I* is a point in the circle *ABE*.

(50.) *To describe three circles of equal diameters, which shall touch each other.*



Take any straight line *AB*, which bisect in *D*; and from the centres *A* and *B*, with the equal radii *AD*, *BD* describe two circles. Upon *AB* describe an equilateral triangle *ABC*, cutting the circles in *E* and *F*; and with the centre *C*, and radius *CE* or *CF*, describe another circle; these circles touch each other as required.

Since $AB = AC$, and *AD* is half of *AB*, ∴ *AE*, which is equal to it, is half of *AC*, and ∴ $AE = EC$. In the same manner $CF = FB$, $\therefore$ the radii of the circles are all equal. And the circles touch each other in D, E, F, because the lines joining the centres pass through those points.

* * *

(51.) *Every thing remaining as in the last proposition; to describe a circle which shall touch the three circles.*

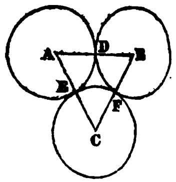
Bisect the angles CAB, CBA (see last Fig.) by the lines AO, BO meeting in O. O is the centre of the circle required.

Join OC. Since the angle $CAB = CBA$, $\therefore$ their halves are equal, or $OAB = OBA$, $\therefore$ $OA = OB$. Also since $CA = AB$, and AO is common, and the angle $CAO = BAO$, $\therefore$ $CO = OB$; and the three lines OA, OB, OC are equal; parts of which GA, HB, CI are equal; $\therefore$ OG, OH, OI are equal; and a circle described from O as a centre, with a radius equal to any one of those lines, will pass through the extremities of the other two, and touch the circles in the points G, H, I; because the lines joining the centres pass through those points.

In nearly the same manner a circle may be described which shall touch the three circles on the opposite circumferences.

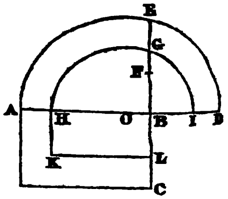
* * *

(52.) *To determine how many equal circles may be placed round another circle of the same diameter, touching each other and the interior circle.*



Let A be the centre of the interior circle, and AD its radius. Describe (vi. 50.) the circles DF, EF touching the circle DE, and each other. Then the angle at A being one third part of two, or one sixth part of four right angles, subtends an arc ED equal to one sixth of the whole circumference. And the same being true of every other contiguous circle, the number of circles which can be described touching each other and the interior one will be six.

(53.) *From a given rectangular parallelogram to cut off a gnomon, whose breadth shall be every where the same, and whose area shall be to that of the parallelogram in any given ratio.*



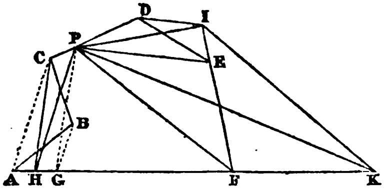
Let AC be the given parallelogram. Produce AB to D making $BD = BC$. On AD describe a semicircle, and produce CB to E; and let the ratio of the part to be cut off, to the whole, be that of $1 : n$. Make $BE : BF :: n : n-1$; and take BG a mean proportional between BE and BF. Bisect AD in O; and with the centre O, and radius OG, describe a semicircle HGI; $AH = ID$, will be the breadth of the gnomon.

Make $BL = BI$, and draw HK, LK parallel to the sides of the parallelogram; then $AC : HL$ in the ratio compounded of the ratios of $AB : BH$ and $BD : BI$, *i. e.* in the duplicate ratio of $BE : BG$, or the ratio of $BE : BF$, *i. e.* in the ratio of $n : n-1$,

$\therefore$ the gnomon AKC is to AC in the ratio of $1 : n$.

(54.) *To describe a triangle equal to a given rectilinear figure, having its vertex in a given point in a side of the figure, and its base in the base (produced if necessary) of the figure.*

Let $ABCDEF$ be the given rectilineal figure, and P a given point in CD, which is to be the vertex of the triangle, the base being in AF. Join CA, and draw BG parallel to it; join CG, PG, PF, PE. Draw CH parallel to PG. Join PH. Draw DI parallel to PE, meeting FE produced in I. Join PI; and draw IK parallel to PF, meeting AF in K. Join PK; HPK will be equal to $ABCDEF$.

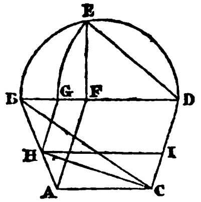


Since BG is parallel to CA, the triangles BAG, BCG are equal; the figure therefore is equal to $GCDEF$. And since GP is parallel to CH, the triangles GCP, GHP are equal. Again since DI is parallel to PE, the triangles PIE, PDE are equal; $\therefore$ $PDEF$ is equal to the triangle PIF, *i. e.* to the triangle PKF, since IK is parallel to PF; whence the whole figure

ABCDEF is equal to the triangles *PHG*, *PGF*, *PKF*, *i. e.* to the triangle *PHK*.

* * *

(55.) *On the base of a given triangle, to describe a quadrilateral figure equal to the triangle, and having two of its sides parallel, one of them being the base of the triangle; and one of its angles being an angle at the base, and the other equal to a given angle.*



Let *ABC* be the given triangle, *AC* its base. At the point *C* make the angle *ACD* equal to the given angle; and let *CD* meet *BD* drawn parallel to *AC*, in the point *D*. On *BD* describe a semicircle *BED*; draw *AF* parallel to *CD*, and *FE* perpendicular to *BD*; and with the centre *D*, and radius *DE*, describe the arc *EG*. Draw *GH* parallel to *AF*, and *HI* to *AC*; *AHIC* will be the figure required.

Join *HC*. Since $DG = DE$,

$$BD : DG :: DG : DF,$$

$\therefore$ (Eucl. v. 19.) $BG : GF :: DG : DF :: HI : AC.$

Now $BG : GF :: BH : HA,$

$$\therefore BH : HA :: HI : AC.$$

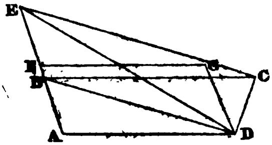
But the triangles *HCI*, *AHC* are in the proportion of $HI : AC$, and the triangles *BHC*, *AHC* in the proportion of $BH : HA$,

$$\therefore HCI : AHC :: BHC : AHC,$$

or $HCI = BHC$; $\therefore$ *ACH*, and *HCI* together are equal to *ACH*, and *BCH* together, or $AHIC = ABC$.

* * *

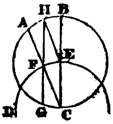
(56.) *A trapezium being given, two of whose sides are parallel; to describe on one of those sides another trapezium, having its opposite side also parallel to this, and one of the angles at the base the same as the former, and the other equal to a given angle.*



Let *ABCD* be the given trapezium whose sides *AD*, *BC* are parallel. Join *BD*; and draw *CE* parallel to it, meeting *AB* produced in *E*. Then the triangles *BCD*, *BED* are equal; and ∴ the triangle *AED* is equal to *ABCD*. Hence (vi. 55.) a figure *ADGF* may be described equal to *ADE*, and ∴ to *ADCB*.

* * *

(57.) *If with any point in the circumference of a circle as centre, and distance from its centre as radius, a circular arc be described; and any two chords be drawn, one from the centre of the circular arc, and the other through the point where this cuts the arc, and parallel to the line joining the centres; the segments of each chord intercepted between the circumferences which are concave to each other, will be equal respectively to those of the other between the other circumferences.*

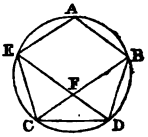


With any point *C* in the circumference of the circle *ABC* as centre, and radius *CE* equal to the distance from the centre *E*, let a circle *DFE* be described. Join *CE*, and draw any chord *CFA*; and through *F* draw *HFG* parallel to *CE*; then will $CF=FH$, and $GF=FA$.

Produce *CE* to *B*, and join *HE*. And since *HG* is parallel to BC, the angle FHE is equal to HEB. Also since the circles are equal, the arc HB is equal to the arc FE (ii. 1. Cor. 1.), $\therefore$ the angle HEB is equal to FCE, $\therefore FCE = FHE$, and HC is a parallelogram; whence $HF = EC = CF$. Also since the rectangle CF, FA is equal to the rectangle HF, FG, and $HF = CF$, $\therefore FA = FG$.

Cor. Hence if any number of lines be drawn parallel to BE, and terminated by the two circumferences, each of them will be equal to BE.

(58.) *If two diagonals of an equilateral and equiangular pentagon be drawn to cut one another, the greater segments will be equal to the side of the pentagon; and the diagonals cut one another in extreme and mean ratio.*



Let $ABDCE$ be an equilateral and equiangular pentagon; draw the diagonals ED, BC cutting each other in F; EF and FB will be each equal to a side of the pentagon; and ED, BC are cut at F, in extreme and mean ratio.

About the pentagon describe a circle. And since $AB = CE$, the arcs AB, CE are equal; $\therefore AE$ is parallel to BC. For the same reason, AB is parallel to EF; $\therefore$ the figure $ABFE$ is a parallelogram; whence $AB = FE$, and $AE = FB$; but $AB = AE$, $\therefore EF = FB$, and each is equal to a side of the pentagon.

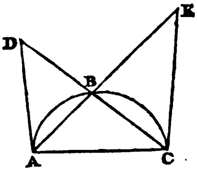
Also the angle $DCF = CDF = DEC$, $\therefore$ the triangles DCF, DEC are similar,

and *ED* : *CD* :: *CD* : *DF*,
or *ED* : *EF* :: *EF* : *FD*;

∴ *ED* is cut in extreme and mean ratio. The same may also be proved of *BC*.

* * *

(59.) *If the sides of a triangle inscribed in the segment of a circle be produced to meet lines drawn from the extremities of the base, forming with it angles equal to the angle in the segment; the rectangle contained by these lines will be equal to the square described on the base.*



Let the sides *AB*, *CB* of the triangle *ABC*, inscribed in the segment *ABC*, be produced to meet *CE*, *AD*, which make with *AC*, angles equal to the angle *ABC* in the segment; the rectangle *AD*, *CE* is equal to the square of *AC*.

Since the angle *ABC*=*DAC*, and the angle at *C* is common to the triangles *ABC*, *ADC*, the triangles are similar. In the same manner it may be shewn that *ABC*, *AEC* are similar; and ∴ *ADC*, *AEC* are also similar; whence

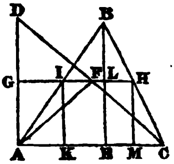
AD : *AC* :: *AC* : *CE*,

and the rectangle *AD*, *CE* is equal to the square of *AC*.

* * *

(60.) *If two triangles (one of them right angled) have the same base and altitude, and the hypothenuse intersect a line which is drawn bisecting the right angle; a*

line passing through this point of intersection parallel to the base, and terminated by the sides of the other triangle, shall be a side of the square inscribed within it.



Let ADC be a right-angled triangle, and ABC on the same base, have its altitude $BE=AD$; and let the hypothenuse DC, meet AF which bisects the angle DAC in F; through which draw GH parallel to AC; IH will be the side of a square inscribed in the triangle ABC.

From I draw IK perpendicular to AC; then (Eucl. vi. 3.)

$$DA : AC :: DF : FC :: DG : (GA=)\,IK,$$

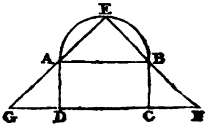
$$\text{also } AC : BE :: IH : (BL=)\,DG,$$

$$\therefore \textit{ex æquo}\ DA : BE :: IH : IK;$$

But $DA=BE$, $\therefore$ $IH=IK$; and if HM be drawn perpendicular to AC, IM is a parallelogram, whose sides are equal; and the angles at K and M being right angles (Eucl. i. 46. Cor.) it is a square.

* * *

(61.) *If on the side of a rectangular parallelogram as a diameter, a semicircle be described, and from any point in the circumference lines be drawn through its extremities to meet the opposite side produced; the altitude of the parallelogram will be a mean proportional between the segments cut off.*

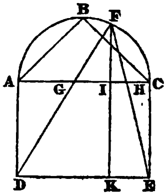


On AB, the side of the rectangular parallelogram $ABCD$, let a semicircle AEB be described; and from any point E, draw EA, EB, and produce them to meet CD produced; AD will be a mean proportional between GD and CF.

Since DG is parallel to BA, the angle DGA is equal to BAE, and the angles at D and E are right angles, $\therefore$ the triangles BAE, DGA are equiangular. In the same manner it may be shewn that FCB is equiangular to BAE, and $\therefore$ to DGA; whence

$$GD : DA :: (CB =)\ DA : CF.$$

(62.) *If on the diameter of a semicircle a rectangular parallelogram be described, whose altitude is equal to the chord of half the semicircle, and lines drawn from any point in the circumference to the extremities of the base intersect the diameter; the squares of the distances of each point of section from the farthest extremity of the diameter will be together equal to the square of the diameter.*



Let ABC be a semicircle, on the diameter of which describe the rectangular parallelogram AE, whose side AD is equal to AB a chord of half ABC; and from any point F in the semicircle draw FD, FE cutting the diameter in G and H; the squares of AH and CG are together equal to the square of AC.

Draw the perpendicular FK; the triangles DGA, DFK being similar,

$$DA : AG :: FK : KD,$$

and ECH, FKE being similar,

$$(CE =)\ DA : CH :: FK : KE,$$

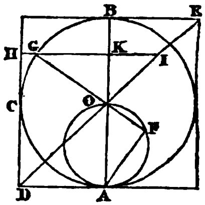
$$\therefore DA^2 : AG \times CH :: FK^2 : (KE \times KD =) IF^2 :: DE^2 : GH^2.$$

Now the square of DE is double of the square of DA, ∴ the square of GH is double of the rectangle AG, CH. But the square of AH is equal to the squares of AG, GH and twice the rectangle AG, GH, *i. e.* to the square of AG and twice the rectangle AG, GC; ∴ the squares of AH and GC are together equal to the squares of AG, GC, and twice the rectangle AG, GC, *i. e.* to the square of AC, (Eucl. ii. 4.).

Cor. The square of the part of the diameter intercepted between the two lines drawn from the point in the semicircle is double of the rectangle contained by the two extreme segments.

* * *

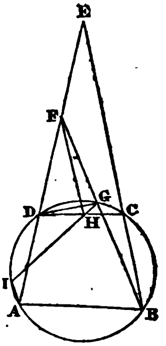
(63.) *If on the radius drawn from the point of contact of a circle and its circumscribed square, another circle be described; and from any point in the outer circumference a line be drawn through its centre to the inner circumference, and through the same point another line be drawn parallel to the common tangent to the circles, and terminated by the side of the square and its diagonal; these two lines are equal.*



Let O be the centre of the circle, circumscribed by a square, whose diagonal is DE. On AO describe a circle AOF; and from any point F draw a line FOG; and through G draw HI parallel to AD; FG is equal to HI.

Join AF; and let HI cut AB in K. Since IG is parallel to AD, it is perpendicular to AB; ∴ the angle GKO is a right angle, and equal to AFO; and the vertical angles at O are equal, and $GO=OA$; ∴ the triangles GKO, OFA are equal, and $OF=OK$. But since $OB=BE$, ∴ (Eucl. vi. 2.) $OK=KI$; and ∴ $OF=KI$, and $OG=KH$; ∴ $FG=IH$.

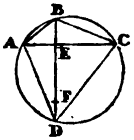
(64.) *If two sides of a trapezium inscribed in a circle be produced, and from the same point in one side produced a line be drawn parallel to the other, intersecting the adjacent side of the trapezium, and a second line to the extremity of that other intersecting the circumference; the line joining the two points of intersection, will pass through the same point.*



Let the two sides AD, BC, of the trapezium $ABCD$ inscribed in the circle ABC, be produced, and let them meet in E; and from any point in AD produced, draw FH parallel to BE, meeting the side DC in H; and join FB, meeting the circumference in G; the line joining G, H will always pass through the same point.

Let GH produced meet the circle in I. Join AI, DG. The angle $GDH=GBC$ in the same segment, and ∴ is equal to the alternate angle GFH; whence a circle may be described through the points G, H, D, F; and ∴ the angle $DGH=DFH=DEB$. But the angle DEB being always the same, DGI, and ∴ DAI, and also the arc DI will be invariable, and D being a fixed point, I must be also; *i. e.* GH will always pass through I.

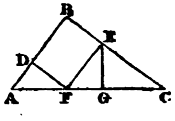
(65.) *If the diagonals of a quadrilateral figure inscribed in a circle cut each other at right angles, the rectangles contained by the opposite sides are together double of the quadrilateral figure.*



Let *ABCD* be a quadrilateral figure inscribed in a circle, whose diagonals *AC*, *BD* cut each other at right angles in *E*; the rectangles contained by *AB*, *CD*, and *AD*, *BC* are together double of the figure.

For (Eucl. vi. *D*.) the rectangles contained by *AB*, *CD*, and *AD*, *BC*, are together equal to the rectangle *AC*, *BD*, *i.e.* to the rectangles contained by *AC*, *BE* and *AC*, *ED*. But the rectangle contained by *AC*, *BE* is double of the triangle *ABC*, and the rectangle contained by *AC*, *ED*, is double of *ADC*; hence the rectangles contained by *AB*, *CD* and *AD*, *BC* are together double of *ABCD*.

(66.) *If a rectangular parallelogram be inscribed in a right-angled triangle, and they have the right angle common; the rectangle contained by the segments of the hypothenuse is equal to the sum of the rectangles contained by the segments of the sides about the right angle.*



Let *ABC* be a right-angled triangle, in which the rectangular parallelogram *DBEF* is inscribed, having one of its angles at *B*; the rectangle *AF*, *FC* is equal to the rectangles *AD*, *DB* and *BE*, *EC* together.

Draw *EG* perpendicular to *FC*. The triangles *ADF*, *EFG* being similar,

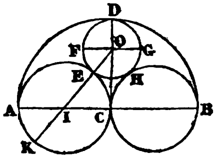
AD : *AF* :: *FG* : (*EF*=) *BD*,

$\therefore$ the rectangle AD, DB is equal to the rectangle AF, FG. In the same manner,

$$AF : (FD =) EB :: EC : CG,$$

$\therefore$ the rectangle BE, EC is equal to the rectangle AF, GC, whence the rectangles AD, DB and BE, EC are together equal to the rectangles AF, FG and AF, GC, *i. e.* to the rectangle AF, FC (Eucl. ii. 1.).

(67.) *If on the diameter of a semicircle, two equal circles be described, and in the curvilinear space included by the three circumferences a circle be inscribed; its diameter will be to that of the equal circles in the proportion of two to three.*



On AB the diameter of the semicircle ADB let two equal circles ACE, BCH be described; and in the curvilinear space let the circle DEG be inscribed; its diameter $FG : AC :: 2 : 3$.

Let O and I be the centres of the circles. Join OI, which will pass through the point of contact E; and produce it to K. From C draw CD perpendicular to AB, which will pass through O. Then the rectangle KO, OE is equal to the square of OC;

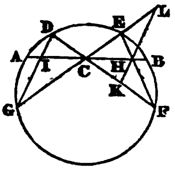
$$\text{and } OE : OC :: OC : OK,$$

$$\therefore OE : OC :: OE + OC : OC + OK :: CD : KE + CD :: 1 : 2;$$

$$\text{and } OE : CD :: 1 : 3,$$

$$\therefore FG : (CD =) AC :: 2 : 3.$$

(68.) *If through the middle point of any chord of a circle two chords be drawn; the lines joining their extremities will intersect the first chord at equal distances from the middle point.*



Let *ACB* be a chord of the circle *ABD*, bisected in *C*; and let *DCF*, *ECG* be any chords drawn through *C*. Join *DG*, *EF* cutting *AB* in *I* and *H*; then will $CI = CH$.

Through *H* draw *KHL* parallel to *DG*, meeting *DF* in *K*, and *GE* produced in *L*. Because *LH* is parallel to *GI*, the angle $HLE = CGI = HFK$, and the vertical angles at *H* are equal, ∴ the triangles *LEH*, *HKF* are equiangular,

$$\therefore LH : HE :: HF : HK,$$

and the rectangle *LH*, *HK* is equal to the rectangle *HE*, *HF*, *i. e.* to the rectangle *AH*, *HB* or the difference of the squares of *AC* and *CH*. The triangles *CID*, *CHK* may in like manner be proved to be equiangular, as also the triangles *CHL*, *CIG*; hence

$$KH : HC :: DI : IC,$$

$$\text{and } LH : HC :: GI : IC,$$

$$\therefore KH \times LH : HC^2 :: DI \times IG : IC^2.$$

But $KH \times LH = AC^2 - HC^2$, and $DI \times IG = AC^2 - IC^2$,

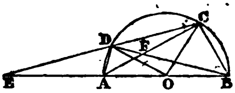
$$\therefore AC^2 - HC^2 : HC^2 :: AC^2 - IC^2 : IC^2$$

$$\textit{comp.}\ AC^2 : HC^2 :: AC^2 : IC^2,$$

$$\therefore HC^2 = IC^2, \text{ and } HC = IC.$$

* * *

(69.) *The longest side of a trapezium being given, and made the diameter of the circumscribed circle; also* *the distance between its extremity and the intersection of the opposite side, produced to meet it; and the angle formed by the intersection of the diagonals; to construct the trapezium.*

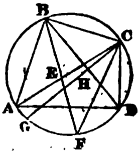


With a diameter equal to the given longest side, describe a circle, and from *O* its centre draw the radii *OC, OD* making with each other an angle equal to twice the complement of the given angle formed by the intersection of the diagonals. Join *CD*, and produce it; and with the centre *O*, and radius equal to the radius of the circle together with the given intercepted distance, describe a circle cutting it in *E*. Join *EO*, and produce it to *B*; *AE* is equal to the given intercepted distance. Join *BC, AD*; *ABCD* is the trapezium required.

Join *AC, BD*. Then the angle *ACB* in a semicircle being a right angle, *FBC* is the complement of *CFB*; but *FBC* is half of *DOC*, and ∴ *BFC* is equal to the given angle to be made by the diagonals.

* * *

(70.) *The diagonals of a quadrilateral figure inscribed in a circle are to one another as the sums of the rectangles of the sides which meet their extremities.*



Let *ABCD* be a quadrilateral figure inscribed in a circle; join *AC, BD*; *AC* is to *BD*, as the rectangles *AB*, *AD* and *CB*, *CD* together, to the rectangles *AB*, *BC* and *AD, DC* together.

Make the angle *ABF* equal to the angle *DBC*; to each of which add the angle *FBD*, ∴ the angle *ABD* is equal to EBC; and ADB is equal to ECB, being in the same segment; $\therefore$ the triangles ABD, EBC are equiangular, and

$$AB : BD :: BE : BC,$$

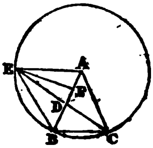
$\therefore$ the rectangle BD, BE is equal to the rectangle AB, BC. Join FC; and since the angle $ABD = FBC$, $\therefore AD = FC$. Also since the angle EFC is equal to BDC in the same segment, and ECF equal to ABF, *i.e.* to DBC; $\therefore$ the triangles ECF, BDC are equiangular,

$$\text{and } BD : DC :: CF : FE,$$

$\therefore$ the rectangle FE, BD is equal to the rectangle CF, CD, *i.e.* to the rectangle AD, DC; whence the rectangles BE, BD and FE, BD or the rectangle BF, BD is equal to the rectangles AB, BC and AD, DC together. In the same manner if the angle BCG be taken equal to the angle ACD, it may be shewn that the rectangle CG, CA is equal to the rectangles AB, AD and CB, CD together. And since the angle $BCG = ACD$, the arc BG is equal to AD, *i.e.* to FC; to each of these add GF; $\therefore$ the arc BAF is equal to GFC, and consequently the line $BF = GC$; $\therefore$ the rectangle AC, CG is equal to the rectangle AC, BF. And $AC : BD$ as the rectangle AC, BF to the rectangle BD, BF, *i.e.* as the rectangles AB, AD and CB, CD together, to the rectangles AB, BC and AD, DC together.

(71.) *The square described on the side of an equilateral and equiangular pentagon inscribed in a circle, is equal to the sum of the squares of the sides of a regular hexagon and decagon inscribed in the same circle.*

Let ABC be an isosceles triangle having each of the



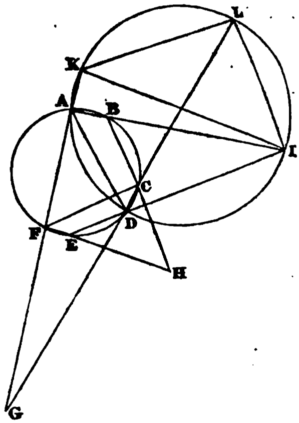
angles at the base double of the angle at *A*. With the centre *A*, and radius *AB*, describe a circle *BCE*. Draw *CE* bisecting the angle *ACB*. Join *EB*, *EA*; and draw *EF* perpendicular to *AB*. Then the angle *EAB* is double of *ECB*, and therefore is equal to *CDB* (Eucl. iv. 10.), and consequently is equal to the vertically opposite angle *ADE*; whence $AE = ED$. Hence *EB*, *ED* and *BC* are equal to the sides of a regular pentagon, hexagon and decagon, respectively inscribed in the circle; and the squares of *BC* and *DE* are together equal to the square of *BE*.

For the angles at *F* being right angles, the squares of *AF*, *FE* are equal to the square of *AE*, *i. e.* to the square of *ED* or to the squares of *EF*, *FD*; whence *AF* is equal to *FD*. And since *AD* is bisected in *F*, and produced to *D*, the rectangle *AB*, *BD* together with the square of *DF* is equal to the square of *BF*; ∴ the rectangle *AB*, *BD* together with the squares of *DF* and *FE*, is equal to the squares of *BF* and *FE*; or the rectangle *AB*, *BD* together with the square of *DE* is equal to the square of *BE*. But (Eucl. iv. 10.) the rectangle *AB*, *BD* is equal to the square of *AD*, *i. e.* to the square of *BC*; ∴ the squares of *BC*, *DE* are together equal to the square of *BE*.

* * *

(72.) *If the opposite sides of an irregular hexagon inscribed in a circle be produced till they meet; the three points of intersection will be in the same straight line.*

Let *ABCDEF* be the hexagon inscribed in the circle; and let its opposite sides meet in *G*, *H*, *I*. Join two opposite angles as A, D; and about ADI describe a circle meeting GA, GD produced if necessary, in K, L. Join FC, KI, LI, LK. Then because $AKID$ is a quadrilateral figure inscribed in a circle, the angle AKI is equal to ADE: and for the same reason ADE is equal to GFE, $\therefore$ the angle AKI is equal to GFE, and KI is parallel to FH. In the same manner it may be shewn that the angle AIL is equal to ADL and consequently to CBI, and LI parallel to HB. Again the angle KLD is equal to DAF, *i. e.* to FCD, and KL is parallel to FC.



$$\text{Hence } GF : FC :: GK : KL,$$
$$\text{and } FC : FH :: KL : KI,$$
$$\therefore GF : FH :: GK : KI;$$

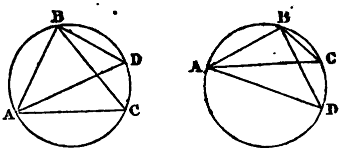
whence G, H, I will be in a straight line.

Triangles in or about circles

SECT. VII.

(1.) *THE vertical angle of any oblique-angled triangle inscribed in a circle, is greater or less than a right angle, by the angle contained by the base and the diameter drawn from the extremity of the base.*

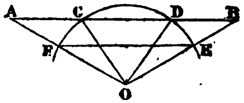
Let ABC be a triangle inscribed in a circle. From A draw the diameter AD; join BD; the angle ABC



is greater or less than a right angle, by the angle *CAD*.

For the angle *ABD* in a semicircle is a right angle; and *ABC* is equal to the sum of *ABD* and *DBC* in one case; and is equal to their difference in the other; and in each case $DBC = DAC$ in the same segment.

(2.) *If from the vertex of an isosceles triangle a circle be described with a radius less than one of the equal sides, but greater than the perpendicular; the parts of the base cut off by it, will be equal.*



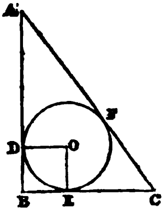
From the vertex *O* of the isosceles triangle *AOB*, with a radius less than *AO*, but greater than the perpendicular from *O* on *AB*, let a circle be described, cutting *AB* in *C* and *D*; $AC = BD$.

Join *EF*, *OC*, *OD*.

$$\text{Then } OE : OB :: OF : OA,$$

and $\therefore$ *EF* is parallel to *AB*; and (ii. 1.) the arc *FC* equal to the arc *DE*, or the angle $FOC = DOE$; but *AO*, *OC* are equal to *BO*, *OD* each to each; $\therefore AC = DB$.

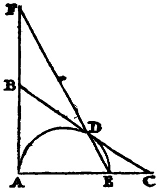
(3.) *If a circle be inscribed in a right-angled triangle; the difference between the two sides containing the right angle and the hypothenuse, is equal to the diameter of the circle.*



Let *DEF* be a circle, inscribed in the right-angled triangle *ABC*. The difference between *AC*, and *AB*, *BC*, is equal to the diameter of the circle.

Find *O* the centre, and join *OD*, *OE*. Then the angles at *D*, *B*, and *E* being right angles, and $OD = OE$, *OB* is a square; and *DB*, *BE* are equal to *OD*, *OE*, *i. e.* are together equal to the diameter of the circle. Now (Eucl. iii. 36. Cor.) $CE = CF$, and $AD = AF$; *i. e.* *AC* is equal to *AD* and *CE*; whence it is less than the sides containing the right angle, by *DB* and *BE*, or by the diameter of the circle.

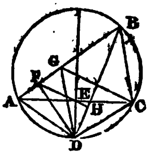
(4.) *If a semicircle be inscribed in a right-angled triangle, so as to touch the hypothenuse and perpendicular, and from the extremity of its diameter a line be drawn through the point of contact to meet the perpendicular produced; the part produced will be equal to the perpendicular.*



Let the semicircle *ADE* touch the hypothenuse *BC* of the right-angled triangle *ABC* in *D*, and the perpendicular in *A*; and from *E* let *ED* be drawn to meet *AB* produced in *F*; $AB = BF$.

Join *AD*. Since *ADE* is a right angle, *ADF* is also a right angle, and ∴ equal to *DAF*, *DFA* together. But *DAF* is equal to *BDA*, since $BD = BA$, being tangents from the same point *B* without the circle; and ∴ the angle $BFD = BDF$, and $BF = BD = BA$.

(5.) *If the base of any triangle be bisected by the diameter of its circumscribing circle, and from the extremity of that diameter a perpendicular be let fall upon the longer side; it will divide that side into segments, one of which will be equal to half the sum, and the other to half the difference of the sides.*



Let the base *BC* of the triangle *ABC* be bisected in *E*, by the diameter of the circumscribing circle *ACD*; and from *D* draw *DF* perpendicular to *AB* the longer side; *BF* will be equal to half the sum, and *AF* to half the difference of *AB*, *BC*.

Join *DA*, *DB*, *DC*; and make $BG = BC$; join *DG*. Since $BG = BC$, and *BD* is common, and the angle $GBD = CBD$, since the arc $AD = DC$; $\therefore DG = DC = DA$, and *DF* is at right angles to *AG*, $\therefore AF = FG$. Whence the sum of *AB* and *BC* is equal to *AG* and 2*BG*, *i. e.* to 2*BF*; and the difference of *AB* and *BC* is equal to the difference of *AB* and *BG*, *i. e.* to 2*AF*.

* * *

(6.) *The same supposition being made, as in the last proposition; if from the point, where the perpendicular meets the longer side, another perpendicular be let fall on the line bisecting the vertical angle; it will pass through the middle of the base.*

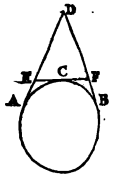
The same construction being made as before; (see last Fig.) let *FH* be drawn perpendicular to *BD*, which bisects the vertical angle; *FH* will pass through *E*.

Because $CB = BG$, and the angle $CBD = GBD$,

∴ *BD* is perpendicular to *CG*; and ∴ *FH* is parallel to *CG*. But since $AF=FG$, ∴ (Eucl. vi. 2.) *AC* is bisected by *FH*; which ∴ passes through *E*.

* * *

(7.) *If a point be taken without a circle, and from it tangents be drawn to the circle, and another point be taken in the circumference between the two tangents, and a tangent be drawn to it; the sum of the sides of the triangle thus formed is equal to the sum of the two tangents.*

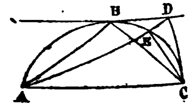


From a given point *D* let two tangents *DA*, *DB* be drawn; and to *C* any point in the circumference between them, let a tangent *ECF* be drawn. The sum of the sides of the triangle is equal to the two tangents *DA* and *DB*.

Since $AE=EC$, and $FC=FB$, ∴ *DE*, *EF*, *FD* together are equal to *AD* and *DB* together. In the same manner, if through any other point in the arc *ACB* a tangent be drawn, it will be equal to the two segments of *DA*, *DB* intercepted between it, and the points of contact *A* and *B*; and the three sides of the triangle so formed will be equal to *DA*, and *DB* together.

* * *

(8.) *Of all triangles on the same base and between the same parallels, the isosceles has the greatest vertical angle.*



Let *ABC* be an isosceles triangle on the base *AC*, and between the parallels *AC*, *BD*. It has a greater vertical angle than any other triangle ADC on the same base, and between the same parallels.

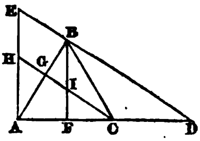
About ABC describe a segment of a circle ABC; and since B is the middle point of the arc, and BD is parallel to AC, BD is a tangent at B. Let the arc cut AD in E; join EC. Then the angle $ABC = AEC$, and ∴ is greater than ADC.

Cor. Of all triangles on the same base and having the same vertical angle, the isosceles is the greatest. For the triangle AEC has the same vertical angle with ABC, and $ABC = ADC$ on the same base and between the same parallels; but ADC is greater than AEC, ∴ ABC is greater than AEC.

* * *

(9.) *If through the vertex of an equilateral triangle a perpendicular be drawn to the side, meeting a perpendicular to the base drawn from its extremity; the line intercepted between the vertex and the latter perpendicular is equal to the radius of the circumscribing circle.*

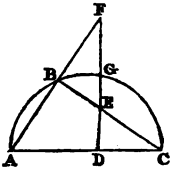
Let BE perpendicular to AB meet AE, which is perpendicular to the base AC, in E; BE is equal to the radius of the circle described about ABC.



Draw BF, CG perpendiculars to the sides; and produce CG to H. Then CI is equal to the radius of the circle described about ABC; and $EBIH$ is a parallelogram. And since CF is equal to FA, (Eucl. vi. 2.) CI is equal to IH, *i.e.* to the opposite side BE; and ∴ BE is equal to the radius of the circumscribing circle.

* * *

(10.) ***If a triangle be inscribed in a semicircle, and a perpendicular drawn from any point in the diameter, meeting one side, the circumference, and the other side produced; the segments cut off will be in continued proportion.***



Let ABC be a triangle in the semicircle ABC; and from any point D in the diameter, let DF be drawn perpendicular to AD, meeting BC, the circumference, and AB produced, in E, G, F;

$$DE : DG :: DG : DF.$$

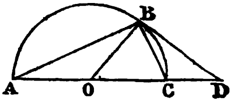
For the angles at E being equal, and the angles at B right angles, ∴ the angle ECD is equal to BFD; and the angles at D are right angles; ∴ the triangles EDC, ADF are similar, and therefore

$$DF : DA :: DC : DE,$$

$$\text{but } DA : DG :: DG : DC;$$

$$\therefore \textit{ex æquo } DF : DG :: DG : DE.$$

(11.) ***If a triangle be inscribed in a semicircle, and one side be equal to the semi-diameter; the other side will be a mean proportional between that side and a line equal to that side and the diameter together.***



Let ABC be a triangle inscribed in the semicircle, and let BC be equal to the semi-diameter; then will

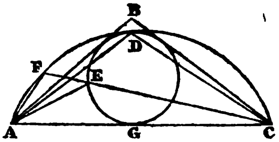
$$BC : BA :: BA : BC+CA.$$

Produce AC to D, making CD equal to the semi-diameter. Take O the centre. Join BD, BO. Since

$BO=BC$, the angle BCO is equal to BOC, *i. e.* to OAB and OBA together, or to $2BAC$. But BCA is equal to CBD and CDB together, *i. e.* to $2CDB$, since $CB=CD$; hence the angle $BAC=BDC$, and $BA=BD$; also the triangles BAD, BCD are similar;
$\therefore BC : (BD=) BA :: BA : AD$, which is equal to BC and CA together.

* * *

(12.) *If a circle be inscribed in a right-angled triangle; to determine the least angle that can be formed by two lines drawn from the extremity of the hypothenuse to the circumference of the circle.*

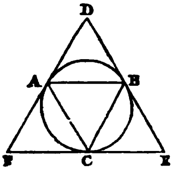


Let ABC be a right-angled triangle, in which a circle DEG is inscribed. On AC describe a segment of a circle ADC, which may touch the inscribed circle in some point, as D. The lines AD, DC, drawn to this point from A and C, contain an angle less than the lines drawn to any other point in the circumference of the circle DEG.

For take any other point E, and join AE, EC; produce CE to F, and join AF. The exterior angle AEC is greater than AFC, *i. e.* than ADC, which is in the same segment. And the same may be proved of lines drawn to every other point in the circumference of the circle DEG.

* * *

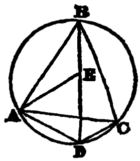
(13.) *If an equilateral triangle be inscribed in a* *circle, and through the angular points another be circumscribed; to determine the ratio which they bear to each other.*



Let ABC be an equilateral triangle inscribed in the circle, about which another DEF is circumscribed, touching the circle in the points A, B, C.

Since DA touches the circle, the angle $DAB = ACB$ (Eucl. iii. 32.); but $ACB = ABC$; $\therefore$ $DAB = ABC$, and they are alternate angles, $\therefore$ DF is parallel to BC. In the same manner it may be shewn that AB is parallel to FE, $\therefore$ $ABCF$ is a parallelogram; and the triangle ABC is equal to AFC. In the same manner ABC may be shewn to be equal to each of the triangles ABD, BCE; and $\therefore$ it is one fourth of the circumscribing triangle.

(14.) *A straight line drawn from the vertex of an equilateral triangle inscribed in a circle to any point in the opposite circumference, is equal to the two lines together, which are drawn from the extremities of the base to the same point.*



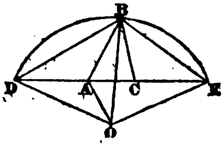
Let ABC be an equilateral triangle inscribed in a circle; from B draw BD to any point D in the circumference. Join AD, CD. BD is equal to AD and CD together.

Make $DE = DA$, and join AE. The angle DAE is equal to the angle DEA; but $ADE = ACB$ in the same segment, $\therefore$ DAE and DEA together are equal to CBA and CAB together; whence DAE

$=CAB$; and taking away the common angle CAE, $DAC = EAB$; but $DCA = EBA$, and $AC = AB$, ∴ $BE = DC$; and BD is equal to AD and CD together.

* * *

(15.) *If the base of a triangle be produced both ways so that each part produced may be equal to the adjacent side, and through the extremities of the parts produced and the vertex a circle be described; the line joining its centre and the vertex of the triangle will bisect the angle at the vertex.*

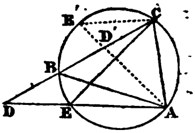


Let AC a side of the triangle ABC be produced both ways till $AD = AB$, and $CE = CB$; and through D, B, E let a circle be described, whose centre is O. If OB be joined, it will bisect the angle ABC.

Join BD, BE, OA, OD, OE. Since $DA = AB$, the angle ABD is equal to ADB; but the angle OBD is equal to ODB, and ∴ the angle OBA is equal to ODA. In the same manner it may be shewn that the angle OBC is equal to OEC; and since ODA is equal to OEC, OBA is equal to OBC; or ABC is bisected by OB.

* * *

(16.) *If an isosceles triangle be inscribed in a circle, and from the vertical angle a line be drawn meeting the circumference and the base; either equal side is a mean proportional between the segments of the line thus drawn.*



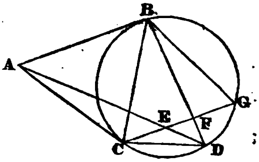
Let ABC be an isosceles triangle inscribed in the circle AEC, the side AB being equal to AC; and from A draw any line AED, meeting the circumference in E, and CB produced in D; AB is a mean proportional between DA and AE.

Join EC. Since $AB = AC$, the angle $ACB = ABC = AEC$ in the same segment; and the angle at A is common to the triangles AEC, ACD, $\therefore$ the triangles are equiangular and similar;

$$\therefore AD : AC :: AC : AE.$$

* * *

(17.) *If from the extremities of one of the equal sides of an isosceles triangle inscribed in a circle, tangents be drawn to the circle, and produced to meet; two lines drawn to any point in the circumference from the point of concourse and one point of contact will divide the base (produced if necessary) in geometrical proportion.*



Let CBG be an isosceles triangle inscribed in a circle, the side CB being equal to BG; and at B and C let tangents BA, CA be drawn, meeting in A. From A and B draw AD, BD to any point D in the circumference, cutting the base CG in E and F;

$$CE : CF :: CF : CG.$$

Join CD. The angle ABC being equal to BGC, is equal to BCG, and $\therefore$ CG is parallel to AB; and the triangles ABC, CBG are equiangular;

$$\therefore BC : CG :: AB : BC;$$

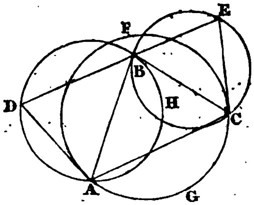
but (vii. 16.) $BF : BC :: BC : BD$,

$\therefore$ *ex æquo* $BF : CG :: AB : BD :: EF : FD$,
since CG is parallel to AB;

$$\text{but } CF : BF :: FD : FG,$$
$$\therefore CF : CG :: EF : FG,$$

whence (Eucl. v. 19.) $CE : CF :: CF : CG$.

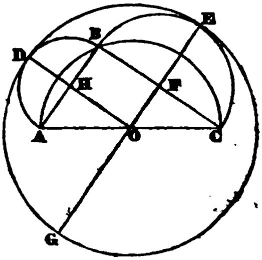
(18.) *If on the sides of a triangle, segments of circles be described similar to a segment on the base, and from the extremities of the base tangents be drawn intersecting their circumferences; the points of intersection and the vertex of the triangle will be in the same straight line.*



On AB, BC, the sides of the triangle ABC, let the segments ADB, BEC be described, similar to AFC the segment on AC. At A and C let tangents AD, CE be drawn. Join DB, BE; they are in the same straight line.

Since DA touches the circle AFG, the angle DAC is equal to the angle in the alternate segment AGC, *i. e.* to the angle in the segment AHB. But the angle ADB, together with the angle in the segment AHB, will be equal to two right angles; $\therefore$ the angles CAD, ADB are equal to two right angles; $\therefore$ AC, DB are parallel. In the same manner AC, BE may be shewn to be parallel; $\therefore$ BD and BE being drawn from the same point, parallel to the same line, will also be in the same straight line.

(19.) *The centre of the circle, which touches the semicircles described on the two sides of a right-angled triangle, is in the middle point of the hypothenuse.*



On the sides *AB*, *BC* of the right-angled triangle *ABC*, let semicircles *ADB*, *BEC* be described. Bisect *AC* in *O*; *O* is the centre of a circle which will touch both the semicircles.

From *O* draw *OFE*, *OHD* perpendicular to the sides. Then *OH* being parallel to *BC*,

(Eucl. vi. 2.) $AO : OC :: AH : HB$,

$\therefore AH = HB$, and *H* is the centre of the semicircle *ADB*. Hence the centre of a circle touching *ADB* in *D* is in the line *DHO*. For the same reason, the centre of a circle touching *BEC* in *E* is in the line *EFO*. Also since *OD* is equal to *OH* and *HD* together, *i. e.* to *BF* and *HB*, or *EF* and *FO* together, *i. e.* to *EO*, *O* is the centre of the circle, which will touch both.

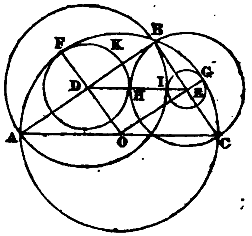
Cor. The diameter of this circle will be equal to the sides of the triangle together.

* * *

(20.) *If on the three sides of a right-angled triangle semicircles be described, and with the centres of those described on the sides, circles be described touching that described on the base; they will also touch the other semicircles.*

On the sides *AB*, *BC* of the right-angled triangle *ABC* let semicircles be described; and with the centres

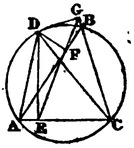
D and *E*, let circles be described touching that described on the base in *F* and *G*; each of these circles will touch the semicircle described on the other side.



Join *ODF*, *OEG*, *DE*. Since *AB*, *BC* are bisected in *D* and *E*, *DE* is parallel to *AC*, and equal to half *AC*, *i. e.* to *AO* or *OC*. In the same manner *OD* is parallel to *BC*, and *OE* to *AB*; ∴ *ODBE* is a parallelogram, and *EB*, *i. e.* $EH = OD$; but $OF = DE$, ∴ $DH = DF$, and *H* is a point in the circumference of the circle *FHK*; and being in the circumference of *BHC*, it will be the point of contact, since *DE* joins the centres. In the same manner it may be shewn that the circle *GI* touches the circle *ABI* in *I*.

* * *

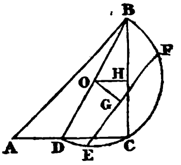
(21.) *If from any point in the circumference of a circle perpendiculars be drawn to the sides of the inscribed triangle; the three points of intersection will be in the same straight line.*



From *D* any point in the circumference of the circle *ABC*, let *DE*, *DF*, *DG* be drawn perpendicular to the sides of the inscribed triangle *ACB*; join *EF*, *FG*; *EFG* is a straight line.

Join *AD*, *BD*, *CD*. Since the angles *DFB*, *DGB* are right angles, a circle may be described about the quadrilateral figure *DGBF* (vi. 13.); and the angle *DFG* is equal to *DBG*. Also since the angles *DFA*, *DEA* are right angles, a circle may be described about the quadrilateral figure $DFEA$; whence the angles DFE, DAE are together equal to two right angles. But $ADBC$ being a quadrilateral figure inscribed in a circle, the angle DAC is equal to DBG, *i.e.* to DFG; $\therefore$ DFE, DFG are equal to two right angles; and EFG is a straight line.

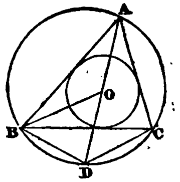
(22.) *The base of a right-angled triangle not being greater than the perpendicular; if on any line drawn from the vertex to the base a semicircle be described, and a chord equal to the perpendicular placed in it, and bisected; the point of bisection will always fall within the triangle.*



Let ABC be a right-angled triangle, of which the side AC is not greater than BC. From B let any line BD be drawn to the base; on which describe a semicircle BCD, and in it place $EF = BC$, which bisect in G; the point G is within the triangle ABC.

Take O the centre of the semicircle; draw OH perpendicular to BC; join OG. Since BC is equal to EF, OH is equal to OG; and the angles at G and H being right angles, a circle described with the centre O, and radius OG, will touch BC in H, and $\therefore$ G is within the angle DBC. Also since AC is not greater than BC, DC is less than BC or EF, $\therefore$ EF is nearer to the centre O, than DC is; or G falls above DC and within the angle DCB.

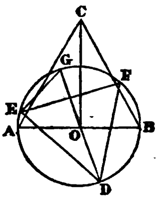
(23.) *The straight line bisecting any angle of a triangle inscribed in a given circle, cuts the circumference in a point, which is equidistant from the extremities of the side opposite to the bisected angle, and from the centre of a circle inscribed in the triangle.*



Let ABC be a triangle inscribed in the circle ACD. Bisect the angles BAC, ABC by the lines AD, BO, which meet in O; O is the centre of the circle inscribed in the triangle. Join BD, DC. The lines DB, DC, DO are equal to each other.

Because the angles DAB, DAC are equal, $BD=DC$; and because the angle $CBD=CAD=DAB$, to each of these add the angle CBO or its equal ABO; and the whole angle OBD is equal to the two ABO, OAB, *i. e.* to BOD (Eucl. i. 32.); and $\therefore$ $OD=DB$.

(24.) *The perpendicular from the vertex on the base of an equilateral triangle is equal to the side of an equilateral triangle inscribed in a circle, whose diameter is the base.*

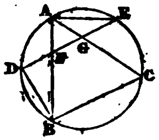


From C the vertex, let CO be drawn perpendicular to AB, the base of the equilateral triangle ABC; upon AB describe a circle ADB, and let DEF be an equilateral triangle inscribed in it; CO will be equal to a side of this triangle.

Draw DG bisecting the angle at D, and $\therefore$ bisecting EF at right angles, consequently passing through the centre. Join EG. The angles ACO, ADG, being each equal to half the angle of an equilateral triangle, are equal to each other, and $AOC=DEG$, each being a right angle, and $AC=AB=DG$, $\therefore$ $CO=DE$.

* * *

(25.) *If an equilateral triangle be inscribed in a circle, and the adjacent arcs cut off by two of its sides be bisected; the line joining the points of bisection will be trisected by the sides.*



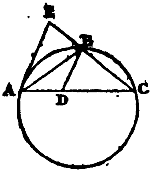
Let ABC be an equilateral triangle inscribed in a circle; bisect the arcs AB, AC in D and E; join DE; it is divided into three equal parts in the points F and G.

Since DE and BC cut off equal arcs BD, CE, they are parallel, and $\therefore$ (Eucl. vi. 2.) $AF=AG$. Join BD, AE. The angle $BFD=AFE$, and $DBF=AEF$ in the same segment, and $BD=AE$, since they subtend equal arcs; $\therefore$ $DF=FA$. In the same manner it may be shewn that $AG=GE$. Now the triangle AFG being similar to ABC is equilateral, $\therefore$ DF, FG, GE are all equal, and DE is trisected.

* * *

(26.) *If any triangle be inscribed in a circle, and from the vertex a line be drawn parallel to a tangent at either extremity of the base; this line will be a fourth proportional to the base and two sides.*

Let ABC be a triangle inscribed in the circle ABC; and from B let BD be drawn parallel to AE a tan-



gent at A; then will $AC : AB :: BC : BD$.

Produce CB to meet the tangent in E. Since the angle EAB is equal to the angle in the alternate segment ACB, and the angle AEB is equal to CBD, ∴ the triangle ABE is similar to CBD,

$$\text{and } AE : AB :: CB : CD;$$

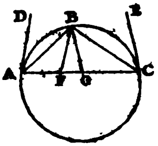
but from similar triangles BDC, EAC,

$$AC : AE :: DC : DB,$$

$$\therefore \textit{ex æquo } AC : AB :: CB : DB.$$

* * *

(27.) *If a triangle be inscribed in a circle, and from its vertex lines be drawn parallel to tangents at the extremities of its base, they will cut off similar triangles.*



Let ABC be a triangle inscribed in a circle, and AD, CE tangents at the points A and C. From B draw BF, BG respectively parallel to them; these lines will cut off the triangles ABF, CBG, which are similar.

For (Eucl. iii. 32.) the angle ACB is equal to DAB, *i. e.* to the alternate angle ABF; and the angle BAC is equal to BCE, *i. e.* to CBG; whence the triangles ABF, CBG having two angles in each equal, will be equiangular and similar.

Cor. 1. The rectangle contained by the segments of the base adjacent to the angles will be equal to the square of either line drawn from the vertex.

For if AD and CE be produced, they will meet and form with *AC* an isosceles triangle, to which *BFG* is similar, ∴ *BF*=*BG*.

Now *AF* : *BF* :: *BG* : *GC*

∴ the rectangle *AF*, *GC* is equal to the rectangle *BF*, *BG*, *i. e.* to the square of *BF*.

Cor. 2. Those segments are also in the duplicate ratio of the adjacent sides.

For the triangles *ABF* and *CBG* are each of them similar to *ABC*,

whence *AC* : *AB* :: *AB* : *AF*,

∴ *AC* : *AF* in the duplicate ratio of *AC* : *AB*;

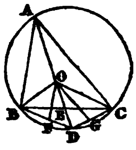
for the same reason,

AC : *CG* in the duplicate ratio of *AC* : *CB*,

∴ *AF* : *CG* in the duplicate ratio of *AB* : *CB*.

* * *

(28.) *If one circle be circumscribed and another inscribed in a given triangle, and a line be drawn from the vertical angle to the centre of the inner, and produced to the circumference of the outer circle; the whole line thus produced has to the part produced the same ratio that the sum of the sides of the triangle has to the base.*



Let *ABD* be a circle circumscribed about the triangle *ABC*; *O* the centre of the inscribed circle. Join *AO*, and produce it to *D*; then *AOD* bisects the angle *BAC*. Join *BD*, *DC*; and draw *BO*, *CO* to the centre of the inscribed circle; then *AD* : *DO* :: *AB*+*AC* : *BC*.

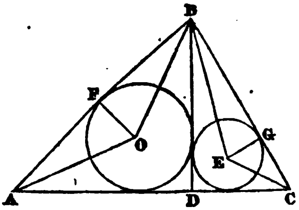
Draw *OF*, *OG* parallel to *AB*, *AC*, meeting *BD*, *CD* in *F* and *G*. The angle *DBC*=*DAC*=*DAB*=*DOF*, and the angle at *D* is common to the triangles

BED, OFD, and (vii. 20.) $BD=DO$, $\therefore OF=BE$. In the same manner it may be shewn that $OG=EC$. Now the trapeziums $BACD$, $FOGD$ being similar, and similarly situated,

$$AD : OD :: AB+AC : FO+OG$$
$$:: AB+AC : BC.$$

* * *

(29.) *If in a right-angled triangle, a perpendicular be drawn from the right angle to the hypothenuse, and circles inscribed within the triangles on each side of it; their diameters will be to each other as the subtending sides of the right-angled triangle.*



Let ABC be a right-angled triangle; from the right angle B let fall the perpendicular BD; and in the triangles ABD, BDC let circles be inscribed; their diameters are to one another as AB to BC.

Bisect the angles BAD, ABD by the lines AO, BO, they will meet in the centre O; in the same manner lines bisecting DBC, DCB meet in the centre E; draw OF, EG to the points of contact. Now the triangles ABD, BDC being similar (Eucl. vi. 8.), $\therefore$ the triangles ABO, BCE are similar; whence

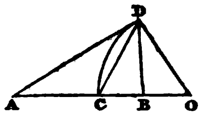
$$AB : BC :: BO : CE;$$

but the triangles OBF, EGC are similar,

$$\therefore BO : CE :: OF : EG :: 2OF : 2EG,$$
$$\therefore AB : BC :: 2OF : 2EG.$$

* * *

(30.) *To find the locus of the vertex of a triangle, whose base and ratio of the other two sides are given.*



Let AB be the given base; divide it in C so that $AC : CB$ may be in the given ratio of the sides. Produce AB to O; and take CO a mean proportional between AO and BO. With the centre C, and radius CO, describe a circle; it will be the locus required.

In the arc AD take any point D; join DA, DB, DC, DO. Since $OD = OC$,

$$AO : OD :: OD : OB,$$

$\therefore$ the sides about the common angle O are proportional, and the triangles ADO, BDO are equiangular;

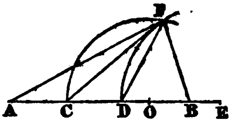
$$\therefore AD : DB :: DO : OB :: CO : OB :: AO : CO$$
$$:: AO - CO : CO - OB :: AC : CB,$$

i. e. in the given ratio. In the same manner, if any other point be taken in the circumference of the circle, and lines drawn to it, they will be in the same given ratio, and $\therefore$ the circumference is the locus required.

Cor. Since in any triangle, if from the vertex a line be drawn cutting the base in the ratio of the sides, it will bisect the angle, $\therefore$ the angle $ADC = BDC$.

(31.) *A given straight line being divided into any three parts; to determine a point such, that lines drawn to the points of section and to the extremities of the line shall contain three equal angles.*

Let AB be the given line, and AC, CD, DB the



given parts. Take CO a mean proportional between AO and DO; and with the centre O and radius OC describe a circle. Produce CB; and make DE a mean proportional between CE and BE; and with the centre E, and radius ED, describe a circle cutting the former in F; F is the point required.

For, as was proved in the last proposition,

$$AF : FD :: AC : CD,$$

and $\therefore$ the angle $AFC = CFD$; and

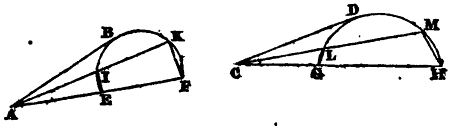
$$CF : FB :: CD : DB,$$

$\therefore$ the angle $CFD = DFB$;

and $\therefore$ the three angles AFC, CFD, DFB are equal.

(32.) *If two equal lines touch two unequal circles, and from the extremities of them lines containing equal angles be drawn cutting the circles, and the points of section joined; the triangles so formed will be reciprocally proportional.*

Let two equal lines AB, CD touch two unequal



circles EBF, GDH; and from A and C let lines AIK, AEF, CLM, CGH be drawn containing the equal angles KAF, MCH. Join IE, KF, GL, MH; then will the triangle $AKF : CHM :: CGL : AIE$.

Since AB is equal to CD, the rectangles EA, AF, and GC, CH are equal;

$$\therefore AF : CH :: GC : AE,$$

and for the same reason,

$$AK : CM :: CL : AI,$$

whence $AF \times AK : CH \times CM :: CG \times CL : AE \times AI$,

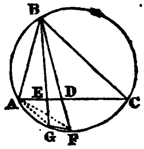
$\therefore$ the triangle $AKF : CMH :: CGL : AIE$,

since $AK : CM$ in the ratio of the perpendiculars from K and M on AF and CH; and $CL : AI$ in the ratio of the perpendiculars from L and I.

* * *

(33.) *If from an angle of a triangle a line be drawn to cut the opposite side, so that the rectangle contained by the sides including the angle, be equal to the rectangle contained by the segments of the side together with the square of the line so drawn; that line bisects the angle.*

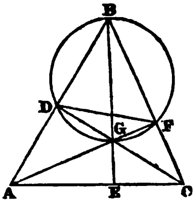
From B one of the angles of the triangle ABC, let BD be drawn, so that the rectangle AB, BC may be equal to the rectangle AD, DC together with the square of BD; BD bisects the angle B.



For if not, let BE bisect it; the rectangle AB, BC is equal to the rectangle AE, EC together with the square of BE. About ABC describe a circle, and produce BD, BE to the circumference in F and G; join FG. The rectangle AD, DC is equal to the rectangle BD, DF; $\therefore$ the rectangle AB, BC is equal to the rectangle BD, DF together with the square of BD, *i. e.* to the rectangle BF, BD. In the same way the rectangle AB, BC is equal to the rectangle BG, BE;

whence the rectangle *BG, BE* is equal to the rectangle *BF, BD*; a circle may therefore be described through *D, E, G, F*; whence *DEG, DFG* are equal to two right angles, *i. e.* to *DEG, DEB*; and ∴ *DFG* is equal to *DEB*, or to *DAB* and *ABG*; and ∴ the arc *AB* equal to the arc *BC*, which is absurd, unless the triangle be isosceles. Hence ∴ *BG* does not bisect the angle; and no other but *BD* can bisect it.

(34.) *In any triangle, if perpendiculars be drawn from the angles to the opposite sides; they will all meet in a point.*



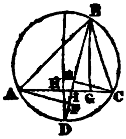
Let *ABC* be any triangle; and *AF, CD* perpendiculars drawn upon the opposite sides, intersecting each other in *G*. Through *G* draw *BGE*; it is perpendicular to *AC*.

Join *FD*; and about the trapezium *BFGD* describe a circle. The triangles *ADG, GFC* being equiangular,

$$AG : GC :: GD : FG,$$

whence also the triangles *AGC, FGD* are equiangular; and ∴ the angle $ACD = DFG = ABE$; and the angle *BAC* is common to the two triangles *ABE, ACD*; ∴ the angle $AEB = ADC$, *i. e.* it is a right angle, and *BE* is perpendicular to *AC*.

(35.) *If from the extremities of the base of any triangle, two perpendiculars be let fall on the line bisecting the vertical angle; and through the points where they* *meet that line, and the point in the base whereon the perpendicular from the vertical angle falls, a circle be described; that circle will bisect the base of the triangle.*



Let *ABC* be a triangle, whose vertical angle *B* is bisected by the line *BD*, on which let fall the perpendiculars *AF*, *CE*. From *B* let fall *BG* perpendicular to *AC*; a circle described passing through *E*, *F*, *G* will also bisect *AC*.

About the triangle *ABC* describe the circle *ADB*; and from *D* draw a diameter which will bisect *AC* in *H*. Now since the angle *AID* is common to the triangles *AIF*, *HID*, and the angles *AFI*, *IHD* are right angles, ∴ the triangles *AIF*, *HID* are similar. In the same manner *BIG*, *CEI* are similar. Whence

$$HI : ID :: IF : IA,$$

$$\text{and } IG : IB :: IE : IC,$$

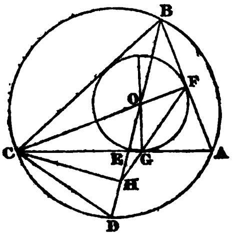
$$\therefore HI \times IG : ID \times IB :: IF \times IE : IA \times IC,$$

and since $ID \times IB = IA \times IC$, ∴ $HI \times IG = IF \times IE$, or a circle passing through *E*, *F*, *G* will pass through *H* (vi. 13.), and ∴ bisect the base *AC*.

* * *

(36.) *If from one of the angles of a triangle a straight line be drawn through the centre of its inscribed circle, and a perpendicular be drawn to this line from one of the other angles; the point of intersection of the perpendicular, and the two points of contact of the inscribed circle, which are adjacent to the remaining angle, are in the same straight line.*

Let *ABC* be a triangle, and *O* the centre of its inscribed circle. From *B* draw *BOD* through the centre;



and from C let fall CH perpendicular to it. Let the circle touch the sides of the triangle in F and G; join HG, GF. HGF is a straight line.

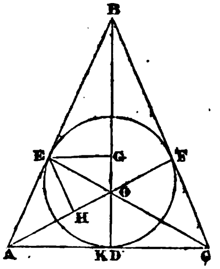
Join OG, OC; and let a circle be described about the triangle ABC; join CD. The triangles OGE, CHE, having the vertical angles at E equal, and OGE, CHE right angles, are similar,

$$\therefore CE : HE :: OE : EG,$$

and $\therefore$ the rectangle CE, EG is equal to the rectangle OE, HE; whence a circle will pass through the points C, O, G, H, $\therefore$ the angle $COH = CGH$. Again (vii. 20.) $CD = DO$, and $AG = AF$, also the angle $CDO = GAF$, $\therefore COE = AGF$; whence $CGH = AGF$; and CG, GA are in the same straight line, $\therefore FG$, GH are in the same straight line.

* * *

(37.) *If from the three angles of any triangle three straight lines be drawn to the points where the inscribed circle touches the sides; these lines shall intersect each other in the same point.*



Let ABC be a triangle, in which a circle is inscribed, touching the sides in E, F, D. Join AF, CE, cutting each other in O. Join BO, and produce it; it will pass through D.

For if not let it pass through some other point K; draw EG, EH respectively parallel to AC, BC. Then the triangles OEH, OFC being similar,

$$OE : EH :: OC : (CF =)\, CD;$$

in the same manner,

$$EG : EO :: KC : CO,$$

$$\therefore EG : EH :: KC : CD.$$

Again, since EH is parallel to BF,

$$AE : EH :: AB : (BF =)\, BE :: AK : EG$$

$$\therefore AK : AE :: EG : EH :: KC : CD,$$

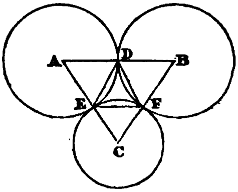
$$\text{or } AK : AD :: KC : CD;$$

$$\therefore AK + KC : AD + DC :: AK : AD;$$

whence $AK = AD$, and K coincides with D.

* * *

(38.) *If three circles touch each other, two of which are equal; the vertical angle of the triangle formed by joining the points of contact, is equal to either of the angles at the base of the triangle, which is formed by joining the centres.*

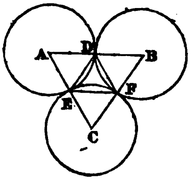


Let three circles, whose centres are A, B, C, touch each other in the points D, E, F; and let the two circles, whose centres are A and B, be equal. Join AB, BC, CA, ED, DF, FE; the angle EDF is equal to either of the angles at A or B.

Since AE is equal to BF, the sides of the triangle ACB are cut proportionally, $\therefore EF$ is parallel to AB, and the angle FED is equal to EDA. Now since CA is equal to CB, the angle at A is equal to the angle at B; but AD, AE are each equal to BF, BD, $\therefore DE$ is equal to DF, and the angle $DFE = FED = EDA = AED$; whence the angle $EDF = DAE = DBF$.

* * *

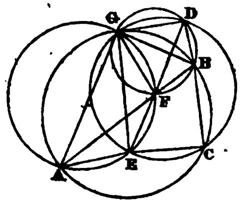
(39.) *If three equal circles touch each other; to compare the area of the triangle formed by joining their centres with the area of the triangle formed by joining the points of contact.*



Let three equal circles, whose centres are *A, B, C,* touch each other in *D, F, E.* Join *AB, BC, CA, ED, DF, FE.*

Since the circles are equal, their radii are equal, ∴ the sides of the triangle *ACB* are cut proportionally, and *DF* is parallel to *AC,* and *DE* to *BC*; ∴ *AEFD* is a parallelogram, and the triangle *DEF* is equal to *ADE.* In the same manner it may be proved to be equal to each of the triangles *DBF, FCE*; and ∴ it is equal to one fourth of *ABC.*

(40.) *If four straight lines intersect each other, and form four triangles; the circles which circumscribe them will pass through one and the same point.*



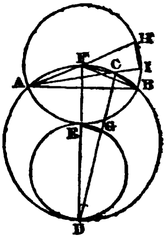
Let the lines *AB, AC, DE, DC* form the four triangles *ABC, AEF, DCE, DBF*; and let the circles circumscribing *AEF, DBF,* intersect each other in *G*; the circles circumscribing the triangles *ABC, DEC* will also pass through *G.*

Join *GA, GE, GF, GB, GD.* Because the points *G, F, B, D* are in the circumference of a circle, the angle $GDB = GFA = GEA$, *i. e.* $GDC = GEA$, and

∴ the points G, E, C, D are in the circumference of the same circle, *i.e.* the circle circumscribing ECD passes through G. Also since the angle $GAE = GFD = GBD$, *i.e.* $GAC = GBD$, ∴ the points G, A, C, B are in the circumference of the same circle; or the circle circumscribing ACB also passes through G.

* * *

(41.) *Having given the base and vertical angle of a triangle; to determine the locus of the extremity of the line which always bisects the vertical angle, and is equal to half the sum of the sides containing the angle.*



Let AB be the given base; and on it describe a segment of a circle ACB, containing an angle equal to the given vertical angle. Complete the circle; and draw the diameter FD bisecting AB. Join AF, FB; and with the centre F, and radius FA, describe a circle ABE, cutting FD in E. On DE as a diameter describe a circle; it will be the locus required.

Let ACB be any position of the triangle, and draw CGD; it bisects the angle at C, since ACD is equal to AFD, *i.e.* to half of AFB or to half ACB. Produce AC, AF to I and H. Join HI, CF, EG. The angle CAF is equal to CDF, and the angles AIH, FCD, DGE are right angles; ∴ the triangles AIH, CDF, EDG are equiangular,

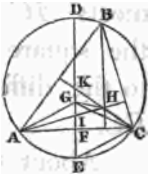
$$\text{and } AH : AI :: FD : CD :: FE : CG.$$

But AH is equal to AF and FB together, *i.e.* to $2FE$, and AI is equal to AC and CB together (ii. 60.), ∴ AC

and CB together are equal to $2CG$, *i. e.* CG is half the sum of the sides AC and CB, and its extremity is in the circumference of the circle EGD.

* * *

(42.) *If from the extremities of the base of a triangle inscribed in a circle, perpendiculars be drawn to the opposite sides, intersecting a diameter which is perpendicular to the base; the segments of the diameter intercepted between these points and a point in it, whose distance from the base is equal to the lesser segment of the diameter made by the base, will be to one another in the ratio of the sides of the triangle.*



Let ABC be a triangle inscribed in a circle, whose diameter DE is perpendicular to the base AC. Make $FG = FE$; and let the perpendiculars be drawn from A and C to the opposite sides, intersecting in H, and meeting the diameter in I and K;

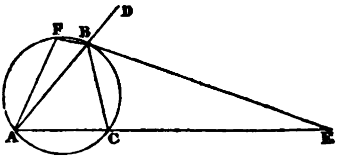
$$KG : IG :: AB : BC.$$

Join AG, GC. Because $GF = FE$, the angles GAF, GCF are each equal to FCE, *i. e.* to half the vertical angle of the triangle; $\therefore$ AGC, ABC are together equal to two right angles (Eucl. i. 32); and since AHC is equal to its vertically opposite angle, AHC, ABC are equal to two right angles; whence $AGC = AHC$; and A, G, H, C are in the circumference of a circle; $\therefore$ the angle $GHA = GCA =$ half the angle ABC. Now the angle KHI, contained by the perpendiculars, is equal to ABC, $\therefore$ GH bisects the angle KHI. Also the angle $GKH = KHB = BAC$; and $KIH = AIF = ACB$;

$\therefore$ the triangle KHI is equiangular to ABC; and it has been shewn that GH bisects the angle KHI,

$$\therefore KG : GI :: KH : HI :: AB : BC.$$

(43.) *If the exterior angle of a triangle be bisected by a straight line which cuts the base produced; the square of the bisecting line is equal to the difference of the rectangles of the segments of the base and of the sides of the triangle.*



Let CBD the exterior angle of the triangle ABC be bisected by BE which meets AC produced in E; the square of BE is equal to the difference of the rectangles AE, EC and AB, BC.

About the given triangle describe the circle ABC; and produce EB to F; and join AF. Then because the angle $EBC = EBD = FBA$, and $AFB = BCE$, since each of them together with ACB is equal to two right angles; $\therefore$ the triangles EBC, FBA are equiangular,

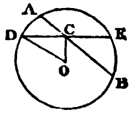
$$\text{and } AB : BF :: EB : BC,$$

$\therefore$ the rectangle AB, BC is equal to the rectangle EB, BF; to each of these equals add the square of BE, and the rectangle AB, BC together with the square of BE is equal to the rectangle EB, BF together with the square of BE, *i. e.* to the rectangle FE, EB, or its equal AE, EC; and $\therefore$ the square of BE is equal to the difference between the rectangles AE, EC and AB, BC.

Squares or Rectangles in connection with a circle.

SECT. VIII.

(1.) *If from the centre of a circle a line be drawn to any point in the chord of an arc; the square of that line together with the rectangle contained by the segments of the chord will be equal to the square described on the radius.*

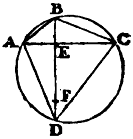


From the centre O of the circle ABD, let OC be drawn to a point C in any chord AB; the square of OC together with the rectangle AC, CB is equal to the square described on the radius.

Through C draw DE perpendicular to OC. Join OD. Then $DC=CE$, and the rectangle DC, CE is equal to the square of DC; but the rectangle DC, CE is equal to the rectangle AC, CB, $\therefore$ the rectangle AC, CB together with the square of CO is equal to the squares of DC, CO, *i. e.* to the square of DO.

* * *

(2.) *If two straight lines in a circle cut each other at right angles; the sums of the squares of the two lines joining their extremities will be equal.*

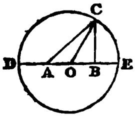


Let the two straight lines AC, BD cut each other at right angles in E; join AB, BC, CD, DA; the squares of AB, CD are equal to the squares of AD and CB.

For the squares of AB and CD are equal to the squares of *AE*, *EB*, *DE*, *EC*. But the squares of *AE* and *DE* are equal to the square of *AD*, and the squares of *EC* and *EB* are equal to the square of *BC*; ∴ the squares of *AB* and *CD* are equal to the squares of *AD* and *BC*.

* * *

(3.) *If two points be taken in the diameter of a circle, equidistant from the centre; the sum of the squares of two lines drawn from these points to any point in the circumference will always be the same.*

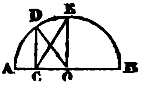


Let *A* and *B* be two points in the diameter of the circle *CDE*, equally distant from the centre *O*; if lines *AC*, *BC* be drawn to a point in the circumference, the sum of the squares of *AC*, *CB* will be the same, in whatever point of the circumference *C* is taken.

Join *CO*; then (iv. 30.) the sum of the squares of *AC*, *CB* is double of the sum of the squares of *AO* and *OC*, which is an invariable quantity.

* * *

(4.) *If from any point in the diameter of a semicircle there be drawn two straight lines to the circumference, one to its point of bisection, and the other at right angles to the diameter; the squares of these two lines are together double of the square of the semi-diameter.*

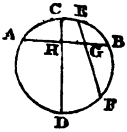


From any point *C* in the diameter *AB*, let *CD*, *CE* be drawn; of which *CD* is perpendicular to *AB*, and *CE* is drawn to the middle point E of the semi-circumference AEB; the squares of CD and CE together will be double of the square of the semi-diameter.

Join DO, OE. The angle EOC is a right angle, and ∴ the square of EC is equal to the sum of the squares of EO and OC; but the square of DC is equal to the difference of the squares of DO and OC; ∴ the squares of EC and CD together are equal to the squares of EO and DO together, *i. e.* are double of the square of EO.

* * *

(5.) *If a straight line be drawn at right angles to the diameter of a circle, and be cut by any other line; the rectangle contained by the segments of this cutting line, together with the square of that part of the perpendicular line which is intercepted between it and the diameter, is always of the same magnitude.*

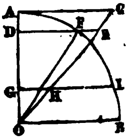


Let AB be drawn at right angles to CD the diameter of the circle ABC; and let it be cut in G, by any other line EF; the rectangle EG, GF, together with the square of HG is of invariable magnitude.

For the rectangle EG, GF is equal to the rectangle AG, GB, and the rectangle AG, GB together with the square of HG is equal to the square of AH, ∴ the rectangle EG, GF together with the square of HG is equal to the square of half AB, which is always the same.

* * *

(6.) *A straight line being drawn from the centre of a quadrant bisecting the arc and meeting a tangent drawn from one extremity; if from any point in the bounding radius a line be drawn parallel to the tangent, the sum of the squares of the segments of it, cut off by the aforesaid line and by the circumference will be equal to the square of the radius.*



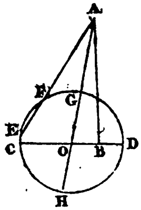
From the centre *O* let *OC* be drawn bisecting the quadrantal arc *AB*, and meeting a tangent to the point *A* in *C*. From any point *D* in *AO* draw a perpendicular *DE*; the squares of *DF* and *DE* are together equal to the square of *OB*.

Join *FO*. Since the angle *DOE* is half a right angle, and the angle at *D* a right angle, ∴ *DEO* is half a right angle, and equal to *DOE*; whence $DE = DO$. Now the squares of *DO* and *DF* are together equal to the square of *OF*; ∴ also the squares of *DE* and *DF* are together equal to the square of *OF*, or *OB*. In the same manner it may be shewn that the squares of *GH* and *GI* are together equal to the square of *OB*.

* * *

(7.) *If from a point without a circle there be drawn two straight lines, one of which is perpendicular to a diameter, and the other cuts the circle; the square of the perpendicular is equal to the rectangle contained by the whole cutting line and the part without the circle, together with the rectangle contained by the segments of the diameter.*

From the point *A* let *AB* be drawn perpendicular

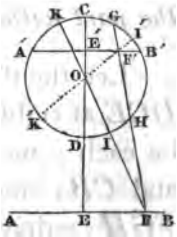


to *CD* the diameter of the circle *DEC*, and *AFE* cutting the circle; the square of *AB* is equal to the rectangles *EA*, *AF*, and *CB*, *BD* together.

Through the centre *O* draw *AGH*. The squares of *AB*, *BO* are equal to the square of *AO*, *i.e.* to the rectangle *HA*, *AG* together with the square of *GO* (Eucl. ii. 6.), *i.e.* to the rectangle *HA*, *AG* together with the rectangle *DB*, *BC* and the square of *OB*; and ∴ the square of *AB* is equal to the rectangles *HA*, *AG* and *DB*, *BC* together.

* * *

(8.) *If any straight line be drawn perpendicular to the diameter of a given circle, and produced to cut any chord; the rectangle contained by the segments of the diameter will be less or greater than the rectangle contained by the segments of the chord, by the square of the line intercepted between them, according as it is drawn without or within the circle.*

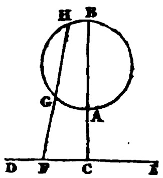


Let *AB* meet the diameter *CD* of the circle *CGD* at right angles in the point *E*, and any other chord *GH* in *F*; the rectangle *CE*, *ED* is less or greater than the rectangle *GF*, *FH*, by the square of *EF*, according as *AB* is without or within the circle.

Take *O* the centre of the circle, and through it draw *FOK* cutting the circle in *I* and *K*. Then because *KI* is bisected in *O*, and produced to *F*, the rectangle *KF*, *FI* together with the square of *OI* is equal to the square of *OF*, *i. e.* to the squares of *OE* and *EF*. But when *E* is without the circle, the rectangle *CE, ED* together with the square of *OD* is equal to the square of *OE*, ∴ the rectangle *KF, FI* together with the square of *OI* is equal to the rectangle *CE, ED*, together with the squares of *OD*, and *EF*. And since *OI* is equal to *OD*, and the rectangle *KF, FI* is equal to the rectangle *GF, FH*, (Eucl. iii. 36. Cor.); ∴ the rectangle *GF, FH* is equal to the rectangle *CE, ED* together with the square of *EF*. In nearly the same manner it is demonstrated if *AB* be within the circle.

* * *

(9.) *If a diameter of a circle be produced to bisect a line at right angles, the length of which is the double of a mean proportional between the whole line through the centre and the part without the circle; and from any point in the double of the mean proportional a line be drawn cutting the circle; the sum of the squares of the segments of the double mean proportional will be equal to twice the rectangle contained by this cutting line and the part without the circle.*



Let the diameter *BA* produced bisect *DCE* at right angles, and let *CD* and *CE* be each a mean proportional between *AC* and *CB*; and through any point *F* let *FGH* be drawn cutting the circle in *G* and *H*; the squares of *DF*, and *FE* are together equal to twice the rectangle *GF, FH*.

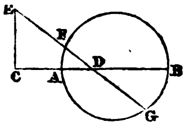
Since the rectangle *AC, CB* is equal to the square of *CD*, the rectangle *AC, CB* together with the square of *CF* is equal to the squares of *CD* and *CF*. But (viii. 8.)

the rectangle *AC, CB* together with the square of *CF* is equal to the rectangle *GF, FH*; whence the rectangle *GF, FH* is equal to the squares of *CD* and *CF* together; and the doubles of equals are equal, ∴ twice the rectangle *GF, FH* is equal to twice the squares of *CD* and *CF* together, *i. e.* to the squares of *DF* and *FE* together (Eucl. ii. 9.).

Cor. If from *F* tangents be drawn to the circle, the sum of their squares will together be equal to the sum of the squares of *DF* and *FE*.

* * *

(10.) *If from a point without a circle two straight lines be drawn, one through the centre to the circumference, and the other perpendicular to it, and on the former a mean proportional be taken between the whole line and the part without the circle; any other line passing through that extremity of the mean proportional which is within the circle, and terminated by the circumference and perpendicular, will be similarly divided.*



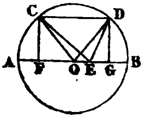
From a point *C* without the circle *ABG*, let *CAB* be drawn through the centre; take a point *D* such that *AC* : *CD* :: *CD* : *CB*; and from *C* let *CE* be drawn perpendicular to *CB*; if through *D*, any line *EFG* be drawn terminated by the circumference and the perpendicular *CE*, *EF* : *ED* :: *ED* : *EG*.

For the rectangle *AC, CB* together with the square of *CE* is equal, by construction, to the squares of *DC, CE*, *i. e.* to the square of *DE*. But (viii. 8.) the rectangle *AC, CB* together with the square of *CE* is equal to the rectangle *FE*, *EG*; ∴ the rectangle *FE*, *EG* is equal to the square of *ED*;

and $EF : ED :: ED : EG$.

* * *

(11.) *If a chord be drawn parallel to the diameter of a circle, and from any point in the diameter lines be drawn to its extremities; the sum of their squares will be equal to the sum of the squares of the segments of the diameter.*



Let *CD* be drawn parallel to *AB* the diameter of the circle *ACD*; and from any point *E* in *AB*, let *EC*, *ED* be drawn; the squares of *EC* and *ED* are together equal to the squares of *EA* and *EB*.

Take *O* the centre, and join *CO*, *DO*; and let fall the perpendiculars *CF*, *DG*. Then since *CD* is parallel to *AB*, the angles *AOC*, *BOD* are equal, and $OF = OG$.

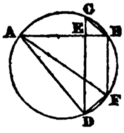
Now (Eucl. ii. 12.) $CE^2 = CO^2 + OE^2 + 2\,OF \times OE$,

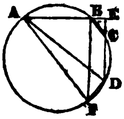
and (Eucl. ii. 13.) $ED^2 = DO^2 + OE^2 - 2\,OG \times OE$,

whence the squares of *CE*, *ED* are equal to twice the squares of *CO*, *OE*, or twice the squares of *AO*, *OE*, *i.e.* to the squares of *AE*, *EB* (Eucl. ii. 9.).

* * *

(12.) *If through a point within or without a circle, two straight lines be drawn at right angles to each other, and meeting the circumference; the squares of the segments of them are together equal to the square of the diameter.*



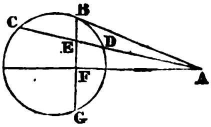


Let *AB*, *CD* cut one another at right angles in *E*; the sum of the squares of *AE*, *EB*, *CE*, *ED* will be equal to the square of the diameter.

Draw the diameter *AF*. Join *FB*, *BC*, *FD*, *DA*; then *ABF* being a right angle is equal to *AED*, and ∴ *BF* is parallel to *CD*, and (ii. 1. Cor. 2.) $BC = FD$. And since the angles at *E* are right angles, the squares of *CE*, *EB* are equal to the square of *CB*, *i. e.* to the square of *DF*; but the squares of *AE*, *ED* are equal to the square of *AD*; ∴ the squares of *CE*, *EB*, *AE*, *ED* are equal to the squares *AD*, *DF*, *i. e.* to the square of *AF*, *ADF* being a right angle.

* * *

(13.) *If from a point without a circle there be drawn two straight lines, one of which touches the circle and the other cuts it; and from the point of contact a perpendicular be drawn to the diameter; the square of the line which touches the circle is equal to the square of that part of the cutting line which is intercepted by the perpendicular, together with the rectangle contained by the segments of that part of it which is within the circle.*

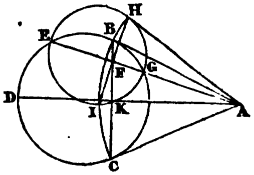


From the point *A* without the circle *BCD* let two lines *AB*, *AC* be drawn; of which *AB* touches the circle, and *AC* cuts it; and from *B* let *BFG* be drawn perpendicular to the diameter; the square of *AB* is equal to the square of *AE* together with the rectangle *CE*, *ED*.

For the square of AB is equal to the squares of AF, FB. But (Eucl. ii. 5.) the square of FB is equal to the rectangle BE, EG together with the square of EF, *i. e.* to the rectangle CE, ED together with the square of EF; $\therefore$ the square of AB is equal to the squares of AF, FE together with the rectangle CE, ED, *i. e.* to the square of AE together with the rectangle CE, ED.

* * *

(14.) *A straight line drawn from the concourse of two tangents to the concave circumference of a circle is divided harmonically by the convex circumference and the chord which joins the points of contact.*



Let AB, AC touch the circle ADC, and AGE cut it. Join BC; then will

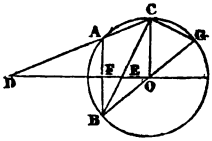
$$AE : AG :: EF : FG.$$

On EG as diameter describe a circle EHG, and through F draw HFI perpendicular to EG. Join AH. Then the rectangle BF, FC is equal to the rectangle EF, FG, *i. e.* to the square of HF, or the rectangle HF, FI; and $\therefore$ the points H, B, I, C are in the circumference of a circle. And since the square of AH is equal to the squares of AF, FH, or to the square of AF and the rectangle BF, FC, *i. e.* to the squares of AK, KF, together with the rectangle BF, FC, *i. e.* to the squares of AK, KB, or to the square of AB; $\therefore$ $AH = AB$; and since the square of AH is equal to the rectangle EA, AG, AH is a tangent at H. And since EG is a diameter, (ii. 42.)

$$AE : AG :: EF : GF.$$

* * *

(15.) *If from the extremities of any chord in a circle straight lines be drawn to any point in the circumference meeting a diameter perpendicular to the chord; the rectangle contained by the distances of their points of intersection from the centre is equal to the square described upon the radius.*



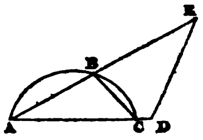
From A and B, the extremities of the chord AB, let AC, BC be drawn to any point C in the circumference; and let them meet a diameter perpendicular to AB in D and E. Take O the centre; the rectangle DO, OE is equal to the square described on the radius.

Draw the diameter BOG. Join CG, CO. Since the angle OCB is equal to OBC, and BGC to FAD, and that CBG, and BGC together are equal to a right angle; ∴ OCE and FAD together are equal to a right angle, and ∴ to FAD and ADF together; hence OCE is equal to ADO, ∴ the triangles COD, COE are equiangular,

$$\text{and } DO : OC :: OC : OE.$$

∴ the rectangle DO, OE is equal to the square of OC.

(16.) *If from any point in the base or base produced, of the segment of a circle, a line be drawn making therewith an angle equal to the angle in the segment, and from the extremity of the base any line be drawn to the former, and cutting the circumference; the rectangle contained by this line and the part of it within the segment is always of the same magnitude.*



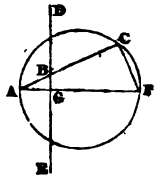
Let ABC be a segment of a circle on the base AC; and from any point D, let DE be drawn, making with AC an angle equal to the angle in the segment, and meeting any line AB drawn from the extremity A; the rectangle EA, AB is of invariable magnitude.

Since the angle ADE is equal to ABC, and the angle at A common to the triangles ADE, ABC, the triangles are ∴ similar; whence

$$AD : AE :: AB : AC,$$

and the rectangle AE, AB is equal to the rectangle AD, AC, which is invariable.

(17.) *To determine the locus of the extremities of any number of straight lines drawn from a given point, so that the rectangle contained by each and a segment cut off from each by a line given in position may be equal to a given rectangle.*



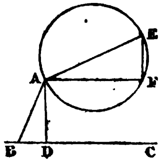
Let A be the given point, and DE the line given in position. Draw AGF perpendicular to DE; and take AF such that the rectangle AG, AF may be equal to the given rectangle; and on AF as diameter describe a circle; it will be the locus required.

Draw any line AC; and join FC. The triangles ABG, AFC being similar,

$$AF : AC :: AB : AG,$$

∴ the rectangle AC, AB is equal to the rectangle AF, AG, *i. e.* to the given rectangle. And the same may be proved of any other line drawn from A to the circumference, which ∴ is the locus.

(18.) *If from a given point two straight lines be drawn, containing a given angle, and such that their rectangle may be equal to a given rectilineal figure, and one of them be terminated by a straight line given in position; to determine the locus of the extremity of the other.*



Let A be the given point, and BC the line given in position. From A draw AD perpendicular to BC; and draw AE, making with it the angle DAE equal to the given angle; and make AE of such a magnitude that the rectangle AD, AE may be equal to the given figure. On AE as diameter describe a circle AFE; it will be the locus required.

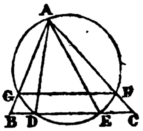
Draw any other line AB, and AF making with it the angle FAB equal to the given angle; join FE. Then the triangles ABD, AFE, being equiangular,

$$AB : AD :: AE : AF,$$

whence the rectangle AB, AF is equal to the rectangle AD, AE, *i. e.* to the given figure; and the same may be proved, of any other two lines, similarly drawn from A.

* * *

(19.) *If from the vertical angle of a triangle two lines be drawn to the base making equal angles with the adjacent sides; the squares of those sides will be proportional to the rectangles contained by the adjacent segments of the base.*



Let AD, AE be drawn from the vertical angle A making equal angles BAD, EAC with the adjacent sides; then will $AB^2 : AC^2 :: BD \times BE : CD \times CE$.

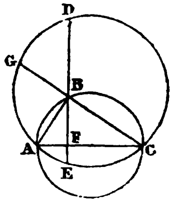
About the triangle *ADE* describe a circle, cutting *AB*, *AC* (produced if necessary) in *G* and *F*. Join *FG*. Then (Eucl. iii. 26.) the arcs *GD*, *FE* are equal, ∴ (ii. 1. Cor.) *FG* is parallel to *BC*;

$$\therefore AB : AC :: BG : CF,$$

$$\text{and } AB^2 : AC^2 :: AB \times BG : AC \times CF$$

$$:: BD \times BE : CD \times CE \text{ (Eucl. iii. 36.).}$$

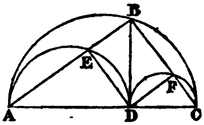
(20.) *If a line placed in one circle be made the diameter of a second, the circumference of the latter passing through the centre of the former; and any chord in the former circle be drawn through this diameter perpendicularly; the rectangle contained by the segments made by the circumference of the latter circle will be equal to that contained by the whole diameter and a mean proportional between its segments.*



Let a line *AC*, placed in the circle *ADC*, be the diameter of the circle *ABC*, whose circumference passes through the centre of *ADC*. Through any point *B* let a line *DBE* be drawn perpendicular to *AC*; the rectangle *DB*, *BE* is equal to the rectangle *AC*, *BF*.

Draw *CBG*. And since the circumference *ABC* passes through the centre of *AGD*, ∴ (ii. 60.) *AB* is equal to *BG*, and the rectangle *AB*, *BC* is equal to the rectangle *GB*, *BC*, *i. e.* to the rectangle *DB*, *BE*. Also the rectangle *AB*, *BC* is equal to the rectangle *AC*, *BF*, (Eucl. vi. C.), ∴ the rectangle *DB*, *BE* is equal to the rectangle *AC*, *BF*.

(21.) *If semicircles be described on the segments of the base made by a perpendicular drawn from the right angle of a triangle; they will cut off from the sides, segments which will be in the triplicate ratio of the sides.*



From the right angle *B* let *BD* be drawn perpendicular to *AC*; and on *AD*, *DC* let semicircles be described, cutting *AB*, *CB* in *E* and *F*; *AE* : *CF* in the triplicate ratio of *AB* : *CB*.

Join *DE*, *DF*; they are perpendicular to *AB*, *BC* respectively; ∴ (Eucl. vi. 8. Cor.)

$$AC : AB :: AB : AD$$
$$AB : AD :: AD : AE;$$

hence *AC* : *AE* in the triplicate ratio of *AC* : *AB*.
In the same manner it may be shewn that

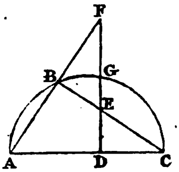
AC : *CF* in the triplicate ratio of *AC* : *CB*,

∴ *inv.* and *ex æquo*,

AE : *CF* in the triplicate ratio of *AB* : *CB*.

* * *

(22.) *If from any point in the diameter of a semicircle a perpendicular be drawn, and from the extremities of the diameter lines be drawn to any point in the circumference, and meeting the perpendicular; the rectangle contained by the segments which they cut off from the perpendicular, will be equal to the rectangle contained by the segments of the diameter.*



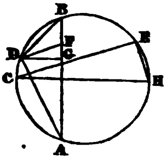
From any point *D* in the diameter *AC* of the semicircle *ABC*, let a perpendicular *DF* be drawn; and to any point *B* in the circumference let the lines *AB*, *CB* be drawn, meeting the perpendicular in E and F; the rectangle FD, DE is equal to the rectangle AD, DC.

Since the angle ABC in a semicircle is a right angle, FBE is also a right angle, and ∴ equal to FDC; and the vertically opposite angles at E are equal, ∴ the angle BFE is equal to ECD, and the triangles FDA, DEC are equiangular,

and $FD : DA :: DC : DE$,

∴ the rectangle FD, DE is equal to the rectangle AD, DC.

(23.) *If from the point of bisection and any other point in a given arc of a circle, two parallel lines be drawn, the former terminated by the circumference, the latter by the chord of the arc; the rectangle contained by these two lines will be equal to that contained by the lines which join the latter point with each extremity of the given arc.*



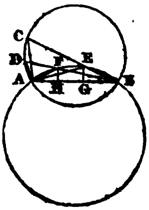
From C the middle point of the arc ACB, and D any other point, let any two parallel lines CE, DF be drawn, of which CE is terminated by the circumference of the circle, and DF by the chord AB. Join AD, DB; the rectangle CE, DF is equal to the rectangle AD, DB.

Draw DG perpendicular to AB; draw the diameter CH; and join EH. The angle FDG being equal to ECH, and the angles at G and E right angles, the triangles FDG, ECH are equiangular,

∴ $DG : DF :: EC : CH$,

whence the rectangle EC, DF is equal to the rectangle DG, CH, *i.e.* (Eucl. vi. C.) to the rectangle AD, DB.

(24.) *If two circles cut each other, and from either point of intersection lines be drawn meeting both circumferences; the rectangles contained by the segments of these lines are to one another in the ratio of the perpendiculars drawn from their intersection with inner circumferences upon the line joining the intersections of the circles.*



Let the two circles *ABC*, *ABE* cut each other in *A* and *B*; join *AB*; and from *B* draw any two lines *BC*, *BD* cutting the circles in *E*, *F*, *C*, *D*; let fall the perpendiculars *EG*, *FH*; the rectangles *BE*, *EC* and *BF*, *FD* are to one another as *EG* to *FH*.

Join *AD*, *AC*. Since the angle $AFB = AEB$, $\therefore$ $AFD = AEC$; but $ADF = ACE$ in the same segment, $\therefore$ the triangles *AFD*, *AEC* are similar;

$$\text{and } EC : EA :: FD : FA.$$

$$\text{But } EC : EA :: EC \times EB : EA \times EB$$

$$= EG \times \text{diameter of the circle } ABE,$$

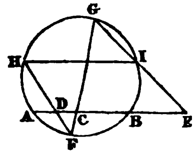
$$\text{and } FD : FA :: BF \times FD : BF \times FA$$

$$= FH \times \text{diameter of the circle } ABE,$$

$$\text{whence } EC \times EB : BF \times FD :: EG \times D : FH \times D :: EG : FH.$$

* * *

(25.) *If on opposite sides of any point in the chord of a circle two lines be taken, one terminating in the chord, the other in the chord produced, whose rectangle is equal to that contained by the segments of the chord; and the extremities of the lines so taken be joined to those* *of any other chord passing through the same point; the line joining their intersections of the circle will be parallel to the first chord.*



On opposite sides of any point C of the chord AB of the circle ABG let two lines CD and CE be taken, such that the rectangle DC, CE may be equal to the rectangle AC, CB; and through C let any chord GCF be drawn, and DF, GE joined, meeting the circumference in H and I. Join HI; it will be parallel to AB.

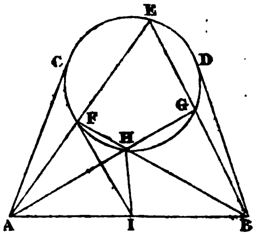
Since the rectangle DC, CE is equal to the rectangle AC, CB, *i. e.* to the rectangle GC, CF,

$$\therefore DC : CF :: GC : CE,$$

$\therefore$ the triangles DCF, GCE are equiangular, and the angle FDC is equal to CGE or FHI in the same segment; $\therefore$ HI is parallel to AB.

(26.) *If from two points without a circle two tangents be drawn, the sum of the squares of which is equal to the square of the line joining those points; and from one of them a line be drawn cutting the circle, and two lines from the other point to the intersections with the circumference; the points in which these two lines cut the circle, are in the same straight line with the former point.*

From A and B two points without the circle CDE let tangents AC, BD be drawn, such that the sum of their squares may be equal to the square of AB. If



from A any line AFE be drawn, and BF, BE joined; the points A, H, G will be in a straight line.

In AB take a point I, so that the rectangle AB, BI may be equal to the square of BD. Join IF, IH, AH, HG. Then the rectangles AB, AI and AB, BI are together equal to the square of AB, *i.e.* to the squares of AC, BD; $\therefore$ the rectangle AB, AI is equal to the square of AC or to the rectangle AF, AE;

$$\therefore AF : AB :: AI : AE,$$

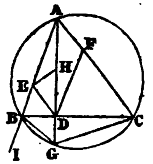
and $\therefore$ (Eucl. vi. 6.) the angle AIF is equal to AEB, whence also $FIB = FHG$. Now the rectangle BI, BA being equal to the square of BD, or to the rectangle BH, BF,

$$\therefore BF : BA :: BI : BH,$$

and $\therefore$ the angle AHB is equal to FIB, or FHG; and BHF is a straight line, $\therefore AHG$ is a straight line.

(27.) *If from the vertex of a triangle there be drawn a line to any point in the base, from which point lines are drawn parallel to the sides; the sum of the rectangles of each side and its segment adjacent to the vertex will be equal to the square of the line drawn from the vertex together with the rectangle contained by the segments of the base.*

From the vertex A of the triangle ABC let a line AD be drawn to any point D in the base; from which let DF, DE be drawn parallel respectively to AB, AC; the rectangles BA, AE, and CA, AF will together be equal to the square of AD, and the rectangle BD, DC together.



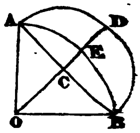
About ABC let a circle be described; and let AD meet the circle in G. Join BG, GC. From E draw EH, making the angle AHE equal to ABG. Produce AB to I. Because the angles AHE, ABG are equal, the points E, B, G, H are in the circumference of a circle, ∴ the rectangle BA, AE is equal to the rectangle GA, AH; and the angle EHD will also be equal to GBI, *i. e.* to ACG. And because AC, DE are parallel, the angle EDH is equal to GAC; hence the triangles EDH, GAC are equiangular,

and ∴ $AC : AG :: DH : DE$,

and the rectangle AG, DH is equal to the rectangle AC, DE, *i. e.* to the rectangle AC, AF. And because the rectangle BA, AE is equal to the rectangle GA, AH and the rectangle CA, AF to the rectangle AG, DH, ∴ the rectangles BA, AE and CA, AF are together equal to the rectangles GA, AH, and GA, DH, *i. e.* to GA, AD or to the rectangle AD, DG together with the square of AD; or to the rectangle BD, DC together with the square of AD.

* * *

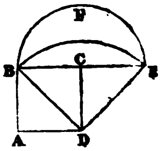
(28.) *If on the chord of a quadrantal arc a semicircle be described; the area of the lune so formed will be equal to the area of the triangle formed by the chord and terminating radii of the quadrant.*



Let ABO be a quadrant, on the chord of which let a semicircle ADB be described; the lune $ADBE$ is equal to the triangle ABO.

Since circles are as the squares of their radii, the quadrant *AEBO* : *ADC* :: $AO^2 : AC^2$ :: 2 : 1, ∴ the quadrant *AEBO* is equal to the semicircle *ADB*; and taking away the part *AEBC*, the lune *ADBE* is equal to the triangle *ABO*.

(29.) *If from the extremities of the side of a square circles be described with radii equal the former to the side, and the latter to the diagonal of the square; the area of the lune so formed will be equal to the area of the square.*

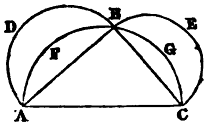


From *D* and *C* the extremities of *DC* the side of a square, with radii *DB* and *CB*, let circles be described, cutting each other again in *E*; the area of the lune *BFE* is equal to the square *AC*.

Join *CE*, *ED*. Since *BC* is equal to *CE*, and *CD* is common, and *BD* is equal to *DE*, ∴ the angles *BCE*, *ECD* are equal; whence *BE* is a straight line; also the angles *BDC*, *EDC* are equal; and *BDC* being half a right angle, *BDE* is a right angle; ∴ the arc *BE* is a quadrant; ∴ the lune *BFE* is equal to the triangle *BDE* (viii. 28.) *i. e.* to the square *AC*.

(30.) *If on the sides of a triangle inscribed in a semicircle, semicircles be described, the two lunes formed thereby will together be equal to the area of the triangle.*

Let *ABC* be a triangle inscribed in a semicircle. On *AB*, *BC* let semicircles *ADB*, *BEC* be described;



the lunes *ADBF*, *BGCE* are together equal to the triangle *ABC*.

Since the areas of circles are as the squares of their diameters, the semicircle $ABC : ADB :: AC^2 : AB^2$,

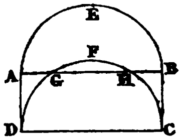
and $ABC : BEC :: AC^2 : BC^2$

$\therefore ABC : ADB + BEC :: AC^2 : AB^2 + BC^2$,

i.e. in a ratio of equality, $\therefore ABC = ADB + BEC$; from these equals take away the segments *AFB*, *BGC*, and the triangle $ABC = AFBD + BGCE$.

* * *

(31.) *If on the two longer sides of a rectangular parallelogram as diameters, two semicircles be described towards the same parts; the figure contained by the two remaining sides of the parallelogram and the two circumferences shall be equal to the parallelogram.*

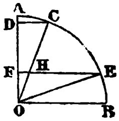


Let *ABCD* be a rectangular parallelogram, on the sides *AB*, *DC* of which let semicircles *AEB*, *DFC* be described; the figure *DAEBCHFG* is equal to *ABCD*.

Since $AB = DC$, the semicircles are equal; from each of which take away *FGH*, and $AGFHBE = DGHC$; if to these equals be added *ADG* and *BHC*, the whole *ADGFHCBE* will be equal to the whole *ABCD*.

* * *

(32.) *If two points be taken at equal distances from the extremities of a quadrant, and perpendiculars be* *drawn from these points to the radius; the mixtilinear space cut off, shall be equal to the sector which stands on the arc between them.*

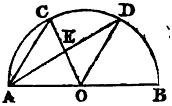


Let two points C, E be taken at equal distances from A and B, the extremities of the quadrant AB; and let fall the perpendiculars CD, EF on AO. Join CO, EO; the figure $CDFE$ is equal to the sector COE.

Since the arc $AC=EB$, the angle $AOC=EOB=OEF$, and the angles at D and F are right angles, and $CO=OE$, ∴ the triangle $COD=EOF$; from each of these take away OFH, ∴ $DFHC=OHE$; to each of these add CHE, and $CDFE=COE$.

* * *

(33.) *If the arc of a semicircle be trisected, and from the points of section lines be drawn to either extremity of the diameter; the difference of the two segments thus made, will be equal to the sector which stands on either of the arcs.*



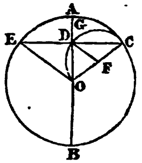
Let the arc of the semicircle ACB be divided into three equal parts in the points C, and D. From A the extremity of the diameter draw AC, AD; take O the centre, and join OC, OD; the difference of the segments AC and ACD is equal to the sector COD.

Since the angles CAD, DAB stand on equal circumferences, they are equal; but the angle $DAO=ODA$, ∴ $CAE=EDO$; and $CEA=OED$, ∴ the triangles CAE, EOD are equiangular; and since OE is drawn bisecting the vertical angle O of the isosceles triangle

AOD, it bisects the base, $\therefore AE=ED$; and consequently the triangle AEC is equal to the triangle DOE; add to each CED, and $CAD=COD$.

* * *

(34.) *If a straight line be placed in a circle, and on the radius passing through one extremity, as a diameter, another circle be described; the segments of the two circles cut off by the above straight line will be similar, and in the ratio of four to one.*



Let EC be a straight line placed in the circle ABC. Take O the centre, and join OC; and upon it describe a semicircle ODC. The segments EAC, DGC are similar, and in the ratio of 4 : 1.

Join OD, and produce it both ways to the circumference. Take F the centre of the semicircle ODC. Join OE, FD. Then ODC being a right angle, $ED=DC$, and $OF=FC$, $\therefore$ (Eucl. vi. 2.) DF is parallel to EO;

$$\text{and } CE : EO :: CD : DF,$$

and the segments EAC, DGC are similar;

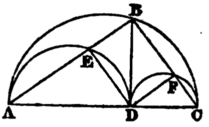
whence $EAC : DGC :: EC^2 : CD^2 :: 4 : 1$.

Cor. The segment ADC is bisected by the circumference DGC.

* * *

(35.) *If on any two segments of the diameter of a semicircle semicircles be described; the area included between the three circumferences will be equal to the*

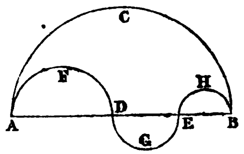
area of a circle, whose diameter is the mean proportional between the segments.



On *AD*, and *DC*, segments of *AC* the diameter of the semicircle *ABC* let semicircles *AED*, *CFD* be described; from *D* draw *DB* perpendicular to *AB*, and ∴ a mean proportional between *AD* and *CD*; the figure *AEDFCB* is equal to the circle described upon *DB*.

Join *AB*, *BC*. Since *ADB* is a right angle, the semicircle on *AB* is equal to those on *AD* and *DB* together; and the semicircle on *BC* is equal to those on *BD* and *DC*; ∴ the semicircles on *AB* and *BC*, or the semicircle *ABC* which is equal to them, will be equal to the semicircles *AED*, *DFC* and the circle described upon *DB*. From these equals take away the semicircles *AED*, *DFC*, and the figure *AEDFCB* is equal to the circle described upon *DB*.

(36.) ***If the diameter of a semicircle be divided into any number of parts, and on them semicircles be described; their circumferences will together be equal to the circumference of the given semicircle.***



Let *AB* the diameter of the semicircle *ACB* be divided into any number of parts in the points *D*, *E*; and on *AD*, *DE*, *EB*, let semicircles be described; their circumferences are together equal to *ACB*.

For since the circumferences of circles are as their diameters,

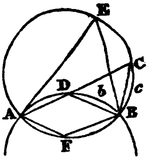
$$ACB : AFD :: AB : AD$$
$$ACB : DGE :: AB : DE$$
$$ACB : EHB :: AB : EB,$$

whence $ACB : AFD + DGE + EHB :: AB : AD + DE + EB$, in which proportion the third term being equal to the fourth,

$$ACB = AFD + DGE + EHB.$$

* * *

(37.) *If two equal circles cut each other, and from either point of section a line be drawn meeting the two circumferences; the area cut off by the part of this line between the two circumferences will be equal to the area of the triangle contained by that part and lines drawn to its extremities from the other point of section.*



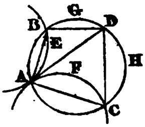
Let the two equal circles ADB, ACB cut each other in A and B; and from A draw any line AC, cutting the circles in D and C; join DB, BC; the figure $DbBcCD$ is equal to the triangle DBC.

Take any points E, F in the circumferences AEB, AFB; join AE, EB, AF, FB. Since the arcs ADB, AFB are equal, the angles ADB, AFB are equal. But the angles AFB, AEB are equal to two right angles, and $\therefore$ to ADB, BDC; whence the angle $BDC = AEB = ACB$, and $BD = BC$; $\therefore$ the segment DbB is equal to the segment BcC; to each of these add $DbBC$, and the triangle DBC is equal to $DbBcCD$.

Cor. If AE is a tangent to ADB at A; the area $ADbBcCEA$ will be equal to the triangle ABE.

* * *

(38.) *If two equal circles touch each other externally, and through the point of contact another be described with the same radius; the area contained by the convex circumferences cut off from the touching circles, and the part of the third without them, is equal to the area of the quadrilateral figure formed by lines drawn from the points of intersection to the point of contact, and to the point where the third circle is cut by a tangent drawn to the point of contact of the two circles.*

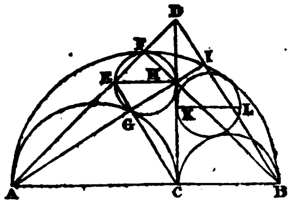


Let two equal circles touch each other in *A*; and through the point of contact let an equal circle *ABC* be described, cutting the former in *B* and *C*. Join *AB*, *AC*; and to the point *A* let a tangent *AD* be drawn; join *BD*, *DC*. The area contained by *AEB*, *AFC* and the intercepted arc *BDC* is equal to the quadrilateral figure *ABDC*.

Since *DA* touches the circle *AEB*, the angle *DAB* is equal to the angle in the alternate segment, and ∴ equal to the angle in the segment *BCA*, *i. e.* equal to the angle *BDA*, whence $BA=BD$; ∴ the segment *BEA* is equal to the segment *BGD*; and *AEBGDA* is equal to the triangle *ABD*. In the same manner it may be shewn that *AFCHDA* is equal to the triangle *ACD*; ∴ *AEBGHCFA* is equal to the quadrilateral figure *ABDC*.

* * *

(39.) *If a straight line be divided into any two parts, and upon the whole and the two parts semicircles be described; and from the point of section a perpendicular* *be drawn, on each side of which circles are described touching it and the semicircles; these circles will be equal.*



Let AB be divided into any two parts in C; and on AB, AC, CB let semicircles be described; from C draw the perpendicular CD, on each side of which let a circle be described touching the perpendicular and each of the semicircles. These circles are equal.

Let $EFGH$ touch the perpendicular in H, and the semicircles in F and G. Draw the diameter EH parallel to AB. Join FE, EA; AF will be a straight line (ii. 35.). Produce it to meet the perpendicular in D. Join FH, HB; FB will also be a straight line, and perpendicular to AD at the point F. Join EG, GC, HG, GA; EC and HA will also be straight lines. Produce AH to I. Join BI; it will be perpendicular to AI, and pass through D, since the perpendiculars to the three sides of the triangle AHB meet in a point. And since the angles AGC, AIB are equal, EC is parallel to DB,

$$\therefore AD : DE :: AB : BC;$$

and AC, HE are parallel,

$$\therefore AD : DE :: AC : EH,$$

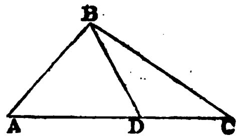
$$\therefore AB : BC :: AC : EH,$$

whence the rectangle AC, CB is equal to the rectangle AB, EH. In the same manner it may be proved that the rectangle AC, CB is equal to the rectangle AB, KL, KL being the diameter of the other circle drawn parallel to AB. Hence $EH = KL$, and the circles are equal.

Construction of Triangles.

SECT. IX.

(1.) *Given one angle, a side adjacent to it, and the difference of the other two sides; to construct the triangle.*



Let *CB* be equal to the given side; draw the indefinite line *CA*, making with it an angle equal to the given angle; and cut off *CD* equal to the given difference. Join *BD*; and make the angle *DBA* equal to *BDA*; *ABC* is the triangle required.

The angle *DBA* being equal to *ADB*, the side *AD* is equal to *AB*; and the difference between *CA* and *AB* is equal to *CD*, *i. e.* the given difference.

* * *

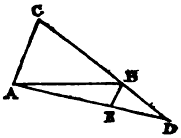
(2.) *Given one angle, a side opposite to it, and the difference of the other two sides; to construct the triangle.*

In any line *CA*, (see last Fig.) take *CD* equal to the given difference; make the angle *CDB* greater than a right angle by half the given angle; from *C* draw *CB* equal to the given side, and meeting *DB* in *B*; and make the angle *DBA* equal to *BDA*; *ABC* is the triangle required.

Since the angles *ABD*, *ADB* are equal, *AB* is equal to *AD*; ∴ the difference between *CA* and *AB* is equal to *CD*, *i. e.* to the given difference. Also the angle at *A* is equal to the difference between the angles *CDB*, *DBA*, or *CDB*, *BDA*, *i. e.* to the given angle. And *CB* was made equal to the given side.

* * *

(3.) *Given the base and one of the angles at the base; to construct the triangle, when the side opposite the given angle is equal to half the sum of the other side and a given line.*



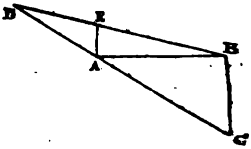
Let *AB* be the given base, and *ABC* the given angle; produce *CB* to *D*, making *BD* equal to the given line. Join *AD*; and from *B* to *AD* draw *BE*, equal to half *BD*. From *A* draw *AC* parallel to *BE*; *ABC* is the triangle required.

For *AB* and *ABC* are made equal to the given base and angle; and since *BE* is parallel to *AC*,

$$AC : CD :: BE : BD :: 1 : 2.$$

* * *

(4.) *Given the base of a right-angled triangle, and the sum of the hypothenuse and a straight line, to which the perpendicular has a given ratio; to construct the triangle.*



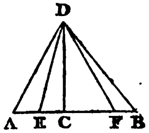
Let *AB* be equal to the given base. From *B* draw *BC* perpendicular to it, and such that it may have to the given sum, the given ratio. Join CA, and produce it; and from B to CD, draw BD equal to the given sum. From A draw AE perpendicular to AB; ABE is the triangle required.

For AE being parallel to BC,

$AE : ED :: BC : BD$, *i. e.* in the given ratio;

$\therefore AE$ is equal to the perpendicular; and AB was made equal to the given base.

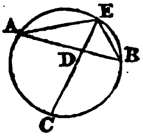
(5.) *Given the perpendicular drawn from the vertical angle to the base, and the difference between each side and the adjacent segment of the base made by the perpendicular; to construct the triangle.*



From any point C in an indefinite line AB, erect a perpendicular CD equal to the given perpendicular; and take CE equal to the given difference between the side and adjacent segment on the opposite side of the perpendicular. Also take CF equal to the other given difference. Join ED, FD; and make the angle $FDA = DFA$, and $EDB = DEB$; ADB is the triangle required.

Since the angles ADF, AFD are equal, $AD = AF$, $\therefore$ the difference between AD, AC is equal to CF, *i. e.* to the given difference. In the same manner the difference between BD and BC is equal to CE, *i. e.* to the given difference.

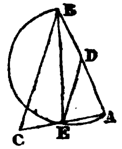
(6.) *Given the vertical angle, and the base; to construct the triangle, when the line drawn from the vertex cutting the base in any given ratio, bisects the vertical angle.*



Let *AB* be equal to the given base; and upon it describe a segment of a circle containing an angle equal to the given angle; and let the base be divided in the given ratio in *D*. Complete the circle; and bisect the arc *ACB* in *C*; join *CD*, and produce it to *E*; join *AE*, *EB*; *AEB* is the triangle required.

For *AEB* is equal to the given angle; and since the arc *AC*=*CB*, the line *ED*, which divides *AB* in the given ratio in *D*, makes the angle *AED*=*DEB*.

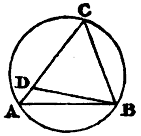
(7.) *Given the vertical angle, and one of the sides containing it; to construct the triangle, when the line drawn from the vertex making a given angle with the base, bisects the triangle.*



Let *AB* be equal to the given side; and on it describe a segment of a circle containing an angle equal to the given angle made by the bisecting line with the base; and make the angle *ABC* equal to the given vertical angle. Bisect *AB* in *D*; and draw *DE* parallel to *BC*, meeting the circle in *E*; join *AE*, and produce it to *C*; *ABC* is the triangle required.

Join *BE*. Since *DE* is parallel to *BC*, (Eucl. vi. 2.) *AE* is equal to *EC*, and ∴ *BE* bisects the triangle; and it makes with the base *AE* an angle equal to the given angle. Also *AB* is equal to the given side, and *ABC* to the given angle at the vertex.

(8.) *Given one angle, a side opposite to it, and the sum of the other two sides; to construct the triangle.*

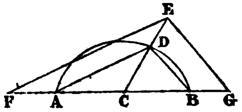


Let AB be the given side. Upon it describe a segment of a circle containing an angle equal to half the given angle; and from A draw AC equal to the given sum of the two sides; join BC; and make the angle CBD equal to BCD; ABD is the triangle required.

Since the angle DCB is equal to DBC, DB is equal to DC; $\therefore$ AD, DB together are equal to the given sum. And the angle ADB is equal to DBC, DCB, *i. e.* to twice DCB, and $\therefore$ is equal to the given angle.

* * *

(9.) *Given the vertical angle, the line bisecting the base, and the angle which the bisecting line makes with the base; to construct the triangle.*



On any line AB describe a segment of a circle containing an angle equal to the given angle. Bisect AB in C; and at C make the angle BCD equal to the given angle which the bisecting line makes with the base; produce it, if necessary, till CE is equal to the given line; draw EF, EG respectively parallel to DA, DB; EFG is the triangle required.

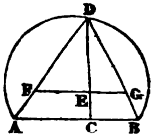
For FE and EG being respectively parallel to DA, DB,

$$FC : CE :: (AC=)\, CB : CD :: CG : CE,$$

$\therefore$ $FC = CG$, and EC, which is equal to the given line, bisects the base; and the angles FEC, CEG being equal to the angles *ADC*, *CDB*, *FEG* is equal to *ADB*, *i. e.* to the given angle.

* * *

(10.) *Given the vertical angle, the perpendicular drawn from it to the base, and the ratio of the segments of the base made by it; to construct the triangle.*



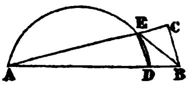
Take any line *AB*, and on it describe a segment of a circle containing an angle equal to the given angle. Divide *AB* in *C*, in the given ratio; and from *C* draw the perpendicular *CD*, from which cut off *DE* equal to the given perpendicular. Join *DA*, *DB*; and through *E* draw *FEG* parallel to *AB*; *DFG* is the triangle required.

Since *FG* is parallel to *AB*,

FE : *EG* :: *AC* : *CB*, *i. e.* in the given ratio, and *DE* is equal to the given perpendicular, and *FDG* to the given angle.

* * *

(11.) *Given the vertical angle, the base, and a line drawn from either of the angles at the base to cut the opposite side in a given ratio; to construct the triangle.*

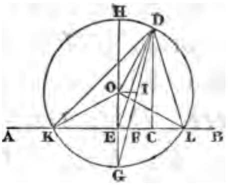


Let *AB* be equal to the given base; and divide it at *D* in the given ratio. On *AD* describe a segment of a circle containing an angle equal to the given angle; and from *B* draw *BE* equal to the given line. Join *AE*, *ED*; and from *B* draw *BC* parallel to *DE*, and meeting *AE* produced in *C*; *ABC* is the triangle required.

For *AB* is the given base; *BE* is the given line; and *AE* : *EC* :: *AD* : *DB*, *i. e.* in the given ratio; and the angle at *C* is equal to *AED*, *i. e.* to the given angle.

* * *

(12.) *Given the perpendicular, the line bisecting the vertical angle, and the line bisecting the base; to construct the triangle.*



From any point *C* in the indefinite line *AB*, draw a perpendicular *CD* equal to the given perpendicular; and with *D* as centre, and radii equal to the two given lines describe circles cutting *AB* in *E* and *F*. Through *E* draw *GEH* perpendicular to *AB*; join *DE*, *DF*; and produce *DF* to meet *HE* in *G*. Bisect *DG* in *I*; and draw *IO* at right angles to *DG*, meeting *GH* in *O*. With the centre *O*, and radius *OG* describe a circle cutting *AB* in *K* and *L*; join *DK*, *DL*; *DKL* is the triangle required.

Join *OD*, *OK*, *OL*. Since *OI* bisects *DG* at right angles, *OD* = *OG*, and the circle passes through *D*. And since *OE* is perpendicular to *KL*, *KE* = *EL*, or *KL* is bisected by *DE*, which is equal to the given bisecting line; and the arc *KG* = *GL*, and the angle *KDF* is equal to *FDL*, or the angle *KDL* is bisected by *DF*, which is equal to the given line; and *DC* was made equal to the given perpendicular.

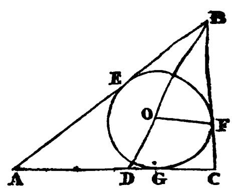
* * *

(13.) *Given the line bisecting the vertical angle,* *the line bisecting the base, and the difference of the angles at the base; to construct the triangle.*

Construct a right-angled triangle *FDC*, (see last Fig.) having its hypothenuse *FD* equal to the given line which bisects the vertical angle, and the angle *FDC* equal to half the given difference of the angles at the base. Produce *FC* both ways; and to it from *D* draw *DE* equal to the given line which bisects the base. Draw *HEG* parallel to *DC*, meeting *DF* produced in *G*. Bisect *GD* in *I*; and from *I* draw *IO* at right angles to *DG*, meeting *GH* in *O*. With the centre *O*, and radius *OG*, describe a circle, cutting *FC* produced in *K* and *L*; join *DK*, *DL*; *DKL* is the triangle required.

For *KE* = *EL*, *i. e.* the base *KL* is bisected by *DE*, which is equal the given line; and the angle *KDF* is equal to *FDL*, being on equal circumferences *KD*, *DL*; *i. e.* the vertical angle *KDL* is bisected by *DF*, which was made equal to the given bisecting line. Also (iii. 5.) the difference between the angles *DLK* and *DKL* is equal to twice the angle *FDC*, *i. e.* to the given angle.

(14.) *Given the vertical angle, and the line drawn to the base bisecting the angle and the difference between the base and the sum of the sides; to construct the triangle.*

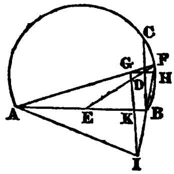


Let *ABC* be equal to the given angle, and *BD* the line bisecting it. Make *BE* and *BF*, each equal to half the given difference. From *F* draw *FO* perpendicular to *BC*, meeting *BD* in *O*. With the centre *O*, and radius *OF* describe a circle, it will touch *AB* in *E* (Eucl. iv. 4.). Through *D* draw a line *AC* touching the circle in *G*; *ABC* will be the triangle required.

For (Eucl. iii. 36. Cor.) *AE*=*AG*, and *GC*=*CF*, ∴ *AC* is equal to *AE* and *CF* together; whence the difference between *AC* and the sum of the sides *AB*, *BC* is equal to *BE*, *BF* together, *i. e.* to the given difference. Also *BD* is equal to the given line, and it bisects the angle *ABC*, which is equal to the given vertical angle.

* * *

(15.) *Given the line bisecting the vertical angle, the perpendicular drawn to it from one of the angles at the base, and the other angle at the base; to construct the triangle.*

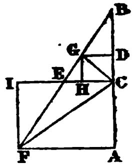


Let *AB* be equal to the given bisecting line; and upon it describe a segment of a circle containing an angle equal to the given angle. Draw *BC* perpendicular to *AB*, and make *BD* equal to the given perpendicular. Bisect *AB* in *E*; join *ED*, and produce it to *F*; join *FA*, *FB*; and through *D*, draw *GDH* parallel to *AB*. In *FB* produced take *BI* equal to *BH*. Join *AI*; *AFI* is the triangle required.

Join *IG*, cutting *AB* in *K*. Because *GH* is parallel to *AB*, and *FE* bisects *AB*, it also bisects *GH*, *i. e.* *GD* = *DH*; but *HB* also is equal to *BI*; ∴ *BD* is parallel to *GI*, and *IK* is half of *IG*, and ∴ equal to *BD* the given perpendicular. Also since *AB* bisects *GI* at right angles, it also bisects the angle *IAG*; and it is equal to the given bisecting line. And *AFI* is equal to the other given angle.

* * *

(16.) *Given the straight line bisecting the vertical angle, and the perpendiculars drawn to that line from the extremities of the base; to construct the triangle.*



On the indefinite straight line *AB*, take *AC*, *CD* respectively equal to the greater and less perpendiculars; and from *C* draw *CE* at right angles to *AB*, and equal to the given bisecting line. Take *AB* : *BD* :: *AC* : *CD*. Join *BE*; and produce it to meet *AF*, *DG*, drawn from *A* and *D*, perpendicular to *AB*. Join *GC*, *CF*; *GCF* is the triangle required.

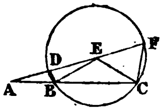
Draw *FI*, *GH* perpendicular to *CE*. Then the triangles *BGD*, *ABF* being similar,

AF : *GD* :: *AB* : *BD* :: *AC* : *CD*,

and the angle at *A* is equal to *GDC*; whence (Eucl. vi. 6.) *AFC*, *GDC* are similar, and the angle *ACF* is equal to *GCD*; and ∴ the angle *FCE* is equal to *ECG*, *i. e.* the angle *FCG* is bisected by *EC*, which was made equal to the given bisecting line. Also *FI*, *GH* are respectively equal to *AC*, *CD*, which were made equal to the given perpendiculars.

* * *

(17.) *Given the vertical angle, the difference of the two sides containing it, and the difference of the* *segments of the base made by a perpendicular from the vertex; to construct the triangle.*



Let *AB* be equal to the given difference of the segments of the base. Draw *BD*, making the angle *ABD* equal to half the given angle; and from *A* draw *AD* meeting it in *D*, and equal to the given difference of the sides; produce it, and make the angle *DBE* = *EDB*, and with the centre *E* and radius *EB* describe the circle *DBC* meeting *AB*, *AD* produced in *C* and *F*; join *EC*: *AEC* is the triangle required.

Join *FC*. Since *BDFC* is a quadrilateral figure inscribed in a circle (Eucl. iii. 22.) the angles *ABD*, *DFC* are equal; but *AEC* is double of *DFC* (Eucl. iii. 20.), and ∴ also of *ABD*, *i. e.* it is equal to the given angle. Also since the angles *EDB*, *EBD* are equal, *ED* = *EB* = *EC*, ∴ *AD* is the difference of the sides *AE*, *EC*, which ∴ is equal to the given difference; and *AB* is evidently equal to the difference of the segments of the base.

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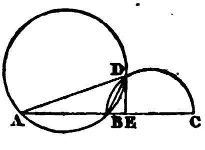
(18.) *Given the base and vertical angle, to construct the triangle, when the square of one side is equal to the square of the base, and three times the square of the other side.*

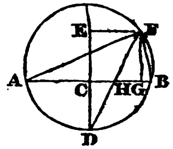


Let *AB* be equal to the given base. Upon it describe a segment of a circle containing an angle equal to the given angle. Produce *AB* to *C*, and make *BC* equal to *BA*; and upon *BC* describe a semicircle

BDC cutting the segment in *D*. Join *AD*, *BD*; *ABD* is the triangle required.

Let fall the perpendicular *DE*; then (Eucl. vi. 8.) the square of *DB* is equal to the rectangle *CB*, *BE* or to the rectangle *AB*, *BE*; and (Eucl. ii. 12.) the square of *AD* is equal to the squares of *AB*, *BD*, and twice the rectangle *AB*, *BE*, *i. e.* to the square of *AB* and three times the square of *BD*. Also, by construction, *ADB* is equal to the given angle.

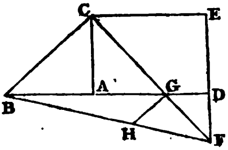
(19.) *Given the base and perpendicular; to construct the triangle, when the rectangle contained by the sides is equal to twice the rectangle contained by the segments of the base made by the line bisecting the vertical angle.*



Let *AB* be equal to the given base; and draw the indefinite line *ED* bisecting it at right angles. Take *CE*, *CD* each equal to the given perpendicular; and through the points *A*, *D*, *B* describe a circle. Draw *EF* parallel to *AB* meeting the circle in *F*. Join *AF*, *FB*; *AFB* is the triangle required.

Draw *FG* perpendicular to *AB*, it is equal to the given perpendicular. Join *FD*. Since *DC* is equal to *CE*, *DF* is equal to twice *DH*; and ∴ the rectangle *DF*, *FH* is double of the rectangle *DH*, *HF*, *i. e.* it is double of the rectangle *AH*, *HB*, contained by the segments of the base, made by *DF* which bisects the angle *AFB*. And *AB* was made equal to the given base.

(20.) *In a right-angled triangle, having given the sum of the base and hypothenuse, and the sum of the base and perpendicular; to construct the triangle.*



Let *BD* be taken equal to the sum of the base and hypothenuse, and *BG* equal to the sum of the base and perpendicular. At the point *G* in the straight line *GB*, make the angle *BGC* equal to half a right angle; and let *CG* be produced to meet a perpendicular to *BD* drawn through *D*, in *F*. Join *BF*; and from *G* to *BF* draw *GH* = *GD*. From *B* draw *BC* parallel to *GH*, meeting *GC* in *C*; and let fall the perpendicular *CA*; *ACB* is the triangle required.

Draw *CE* parallel to *AD*. Because *GH* is equal to *GD*, *BC* is equal to *CE*, *i. e.* to *AD*; and ∴ *BC* and *BA* together are equal to *BD*, the given sum of the hypothenuse and base. And since *AGC* is half a right angle, and the angles at *A* right angles, *AG* is equal to *AC*, ∴ *BA* and *AC* together are equal to *BG*, the given sum of the base and perpendicular.

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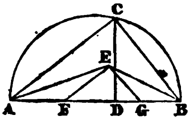
(21.) *Given the perimeter of a right-angled triangle whose sides are in geometrical progression; to construct the triangle.*



Let *AB* be equal to the given perimeter; and on it describe a semicircle *ACB*. Divide *AB* in extreme and mean ratio in *D*; and from *D* draw *DC* at right angles to AB. Join AC, CB. Bisect the angles CAB, CBA by the lines AE, BE, meeting in E; and draw EF, EG respectively parallel to AC, CB; FEG is the triangle required.

Since FE is parallel to AC, the angle $FEA = EAC = EAF$, and $\therefore AF = FE$. And in the same manner it may be shewn that $EG = GB$. Hence GF, FE, EG are together equal to AB the given perimeter. And the angles at F and G being equal to the angles BAC, BCA, the angle FEG is equal to ACB, and is $\therefore$ a right angle.

Also since $AB : AD :: AD : DB$

and (Eucl. vi. 8. Cor.) $AB : BC :: BC : DB$,

$\therefore AD = BC$;

whence also $AB : AC :: AC : (AD =) BC$;

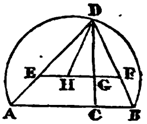
and since the triangles ABC, FGE are equiangular,

$FG : FE :: FE : EG$,

i. e. the sides are in geometrical progression.

* * *

(22.) *Given the difference of the angles at the base, the ratio of the segments of the base made by the perpendicular, and the sum of the sides; to construct the triangle.*



Take any line AB, and divide it in C, in the given ratio of the segments. On AB describe a segment of a circle containing an angle equal to the given difference. From C draw the perpendicular CD. Join AD, DB; and take $DE : DA$ in the ratio of the given sum, to the sum of AD, DB; and through E draw EF parallel to AB; and make the angle GDH equal to GDF, and

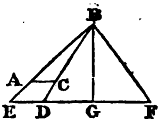
∴ DH equal to DF, and GH equal to GF; DEH is the triangle required.

For $DE : DA :: DF : DB :: DE + DH : DA + DB$, ∴ $DE + DH$ is equal to the given sum. Also the angle EHD is equal to the two HDF, DFH, *i.e.* to HDF, DHF, and DEH is equal to the difference between DHF and EDH; ∴ the difference between EHD and DEH is equal to the sum of HDF and EDH, *i.e.* to EDF, or the given difference.

Also $EG : GH :: EG : GF :: AC : CB$ *i.e.* in the given ratio.

* * *

(23.) *Given the difference of the angles at the base, the ratio of the sides, and the length of a third proportional to the difference of the segments of the base made by a perpendicular from the vertex and the shorter side; to construct the triangle.*



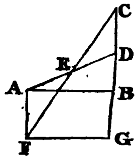
Let ABC be equal to the given difference of the angles at the base; and take $AB : BC$ in the given ratio of the sides. Join AC; and produce BC to D, so that BD may be to the given third proportional in the ratio $CA : CB$. Through D draw EDF parallel to AC; and from B draw BG perpendicular to it; make $GF = GD$; join BF; EBF is the triangle required.

Since $DG = GF$, and the angles at G are right angles; ∴ $BF = BD$, and the angle $BFD = BDF$. Hence the difference between BFE and BEF is equal to the difference between BDF and BEF, *i. e.* to EBD or the given difference.

Also $EB : BF :: EB : BD :: AB : BC$, *i.e.* in the given ratio. And $DE : BF :: DE : BD :: CA : CB :: (BD =) BF$: the given third proportional, whence DE is equal to the given difference of the segments of the base.

* * *

(24.) *Given the base of a right-angled triangle; to construct it, when parts, equal to given lines, being cut off from the hypothenuse and perpendicular, the remainders have a given ratio.*



Let AB be equal to the given base. From B draw BC at right angles to it, and equal to the part to be cut off from the perpendicular. Take CD to the given part to be cut off from the hypothenuse, in the given ratio of the remainders; and from C to DA, draw CE equal to that given part, and produce it. From A draw AF perpendicular to AB, meeting CE produced in F. Draw FG parallel to AB; CFG is the triangle required.

Since AF is parallel to CD, the angles AFE, ECD are equal, and the angles at E are equal, ∴ the triangles AEF, CED are equiangular,

and ∴ $FE : AF :: CE : CD$.

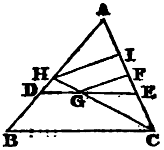
And since AG is a parallelogram, $AF = BG$,

∴ $FE : BG :: CE : CD$,

i.e. in the given ratio of the remainders; and FG is equal to the given base; and CB, CE are equal to the given parts to be cut off.

* * *

(25.) *Given one angle of a triangle, and the sums of each of the sides containing it and the third side; to construct the triangle.*

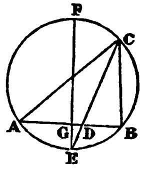


Let BAC be equal to the given angle; and AB, AC equal to the given sums. Join BC; and draw any line DE parallel to it. Make CF, FG, each equal to BD; join CG, and produce it to H; and draw HI parallel to GF. AHI is the triangle required.

Since DG is parallel to BC, and GF to HI, $BD : BH\ (:: CG : CH) :: GF : HI :: CF : CI$, and since BD, GF and FC are equal, BH, HI and IC are also equal; whence AH and HI together are equal to AB; and AI, IH are together equal to AC; and HAI is equal to the given angle.

* * *

(26.) *Given the vertical angle, and the ratio of the sides containing it, as also the diameter of the circumscribing circle; to construct the triangle.*



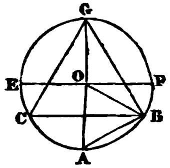
On the given diameter describe a circle; and from it cut off a segment ACB containing an angle equal to the given vertical angle. Divide the base AB in D, in the given ratio of the sides; and draw the diameter EF at right angles to AB. Join ED, and let it meet the circumference in C; join AC, CB; ACB is the triangle required.

Since $AE = EB$, the angle ACB is bisected by CE,

$\therefore AC : CB :: AD : DB$, *i.e.* in the given ratio.

* * *

(27.) *Given the vertical angle, and the radii of the inscribed and circumscribing circles; to construct the triangle.*



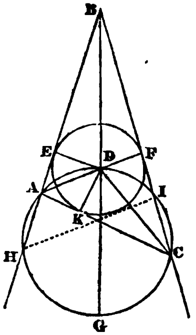
With the given radius describe the circumscribing circle; and from any point A in it take AB, AC each equal to the arc subtending at the centre an angle equal to the given angle. Join BC; and parallel to it, at a distance equal to the radius of the inscribed circle draw EF. Join AB; and from A to EF draw $AO=AB$, and produce AO to G. Join CG, BG; GCB is the triangle required.

For the angle CGB (Eucl. iii. 20.) is equal to the angle at the centre, which stands on AC, and $\therefore$ is equal to the given angle. Also the angle $CGA=AGB$. Join BO; and since $AB=AO$, the angle $AOB=ABO$; but AOB is equal to the two OBG, OGB; and $AGB=ABC$, $\therefore$ $CBO=OBG$, and BO bisects the angle CBG; whence O, the point of intersection of the bisecting lines GA and BO, is the centre of the inscribed circle; and its distance from BC was made equal the given radius.

* * *

(28.) *Given the vertical angle, the radius of the inscribed circle, and the rectangle contained by the straight lines drawn from the centre of that circle to the angles at the base; to construct the triangle.*

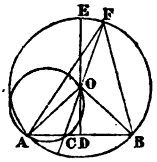
Let the angle ABC be equal to the vertical angle, which bisect by the straight line BD. Let D (ii. 14.) be the centre of a circle which would touch BA and BC;



and let *E* and *F* be the points of contact; join *DE*, *DF*, and produce *BD* to *G*, so that the rectangle *DE*, *DG* may be equal to the given rectangle. On *DG* describe a circle cutting *BA*, *BC* in *A* and *C*; and in *H* and *I*. Join *AC* (or *HI*). *ABC* or *AHI* is the triangle required.

Join *AD*, *DC*; and draw *DK* perpendicular to *AC*. Then since *BA* and *BC* make equal angles with *BG* which passes through the centre, the arcs *AD*, *DI*, which they cut off, are equal; ∴ the angle *ACD* is equal to *DCI*, *i. e.* *ACB* is bisected by *CD*, or *D* is the centre of the circle inscribed in the triangle *ABC*; ∴ *K* is the point of contact, and *DK*=*DE*. But (Eucl. vi. C.) the rectangle *AD*, *DC* is equal to the rectangle *DK*, *DG*, *i. e.* to the rectangle *DE*, *DG*, which is equal to the given rectangle.

(29.) *Given the base, one of the angles at the base, and the point, in which the diameter of the circumscribing circle drawn from the vertex meets the base; to construct the triangle.*



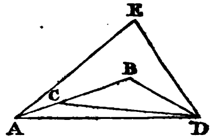
Let *AB* be equal to the given base, and *C* the given point. Bisect *AB* in *D*, and draw *DE* at right angles to *AB*. Upon *AC* describe a segment of a circle containing an angle equal to twice the difference between the given angle and a right angle; and let it meet *DE* in *O*. Join *AO*, *OC*; and produce

CO to *F*, making *OF* = *OA*; join *AF*, *FB*; *AFB* is the triangle required.

Join *OB*. Because *AD* = *DB*, and *DO* is common, and the angles at *D* right angles, ∴ *OB* = *OA* = *OF*; and a circle described from *O* as a centre, and radius *OA*, will pass through *F* and *B*, and circumscribe the triangle *ABF*. And *FOC* produced is a diameter. Also the angles *AOC*, *AOF* are equal to two right angles, *i. e.* to *AOC* and twice the given angle; ∴ *AOF* is equal to twice the given angle; and since it is double of *ABF*, *ABF* will be equal to the given angle.

* * *

(30.) *Given the vertical angle, the base, and the difference between two lines drawn from the centre of the inscribed circle to the angles at the base; to construct the triangle.*



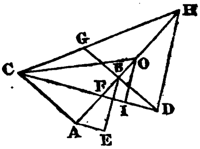
Take any line *AB*, unlimited towards *B*, and cut off *AC* equal to the given difference; and at the point *C* make the angle *BCD* such that its quadruple together with the given angle at the vertex may be equal to two right angles, From *A* to *CD*, draw *AD* equal to the given base. At the point *A* make the angle *BAE* = *BAD*, and at the point *D* make the angle *CDB* = *DCB*, and *BDE* = *BDA*; *EDA* is the triangle required.

For the angles *EAD*, *EDA* being bisected by *AB*, *BD*, *B* is the centre of the inscribed circle. And the angle *BCD* being equal *BDC*, *BC* is equal to *BD*, and

$\therefore$ the difference between BA and BD is equal to AC, *i. e.* to the given difference. Also the angles EAD, EDA are together double of the angles BAD, BDA, *i. e.* of BCD and BCD, and $\therefore$ are equal to $4\,BCD$; whence the angle AED together with $4\,BCD$ will be equal to two right angles; and $\therefore$ the angle AED is equal to the given vertical angle. Also AD is equal to the given base.

* * *

(31.) *Given that segment of the line bisecting the vertical angle which is intercepted by perpendiculars let fall upon it from the angles at the base; the ratio of the sides; and the ratio of the radius of the inscribed circle to the segment of the base which is intercepted between the line bisecting the vertical angle and the point of contact of the inscribed circle; to construct the triangle.*



Let AB be equal to the given segment of the bisecting line; from A and B, on opposite sides, draw AC, BD at right angles to it. On AB, as hypothenuse, describe a right-angled triangle AEB, whose sides are in the given ratio of the radius of the inscribed circle to the segment of the base intercepted between the bisecting line and the point of contact of the inscribed circle. Divide AB in F in the ratio of the sides; and through F, draw CFD parallel to AE, and meeting the perpendiculars in C and D. Make $BG = BD$. Join CG, and produce it to meet AB produced in H; join HD; HCD is the triangle required.

Draw CO bisecting the angle HCD. Since $BG = BD$, and BH is perpendicular to GD, it bisects the angle *GHD*; whence *O* is the centre of the inscribed circle. From *O* draw *OI* perpendicular to *CD*, and ∴ parallel to *BE*; whence

$$OI : IF :: BE : EA.$$

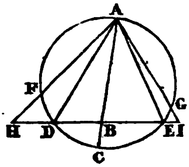
i. e. in the given ratio of the radius of the inscribed circle to the segment of the base intercepted between the bisecting line and the point of contact of the inscribed circle. Also from the similar triangles *AFC*, *BFD*,

$$AF : FB :: CF : FD :: CH : HD,$$

$$\therefore CH : HD \text{ in the given ratio of the sides.}$$

* * *

(32.) *Given the line bisecting the vertical angle, and the differences between each side and the adjacent segment of the base made by the bisecting line; to construct the triangle.*

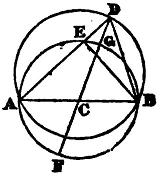


Let *AB* be equal to the given bisecting line. Produce it to *C*, so that *BC* may be a fourth proportional to *AB* and the given differences. On *AC*, as diameter, describe a circle *ADE*; in which place a line *DBE*, passing through *B*, equal to the sum of the given differences; and from *A* draw *AF*, *AG* to the circumference, each equal to *DE*; and produce them to meet *DE* produced in *H* and *I*; *AHI* is the triangle required.

Since *DE* is equal to the sum of the given differences, and the rectangle *DB*, *BE*, is equal to the rectangle *AB*, *BC*, *i. e.* to the rectangle contained by the given differences, *DB* and *BE* will be the given differences. And since *AF*=*AG*, the arcs *AF*, *AG* are equal and AC being a diameter, FC and CG will also be equal, $\therefore$ the angles FAC, CAG are equal, or HAI is bisected by AC. Also since $FA = DE$, AH and HE will be equal; $\therefore$ the difference between AH and HB is equal to BE one of the given differences. In the same manner, $AI = ID$; and $\therefore$ the difference between AI and IB is equal to BD, the other given difference.

* * *

(33.) *Given one of the angles at the base, the side opposite to it, and the rectangle contained by the base and that segment of it made by the perpendicular which is adjacent to the given angle; to construct the triangle.*

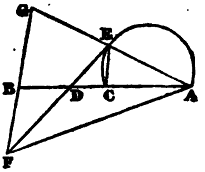


Let AB be equal to the given side. Upon it describe a segment of a circle containing an angle equal to the given angle. Bisect AB in C; and from C draw to the circumference, a line CD such that the difference between the squares of CD and CB may be equal to the given rectangle. Join AD, DB; ADB is the triangle required.

On AB describe a circle ABE. Join BE. Then the rectangle AD, DE is equal to the rectangle GD, DF, *i. e.* to the difference of the squares of CD and CG or to the given rectangle. And BE is perpendicular to AD; AB is equal to the given side, and ADB to the given angle.

* * *

(34.) *Given the vertical angle, and the lengths of two lines drawn from the extremities of the base to the points of bisection of the sides; to construct the triangle.*

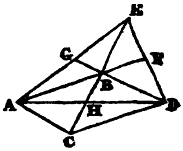


Let *AB* be equal to one of the given lines; bisect it in *C*; and on *AC* describe a segment of a circle containing an angle equal to the given angle. From *BA* cut off *BD* equal to one third of *BA*; and from *D* to the circle, draw *DE* equal to one third of the other given line. Produce *ED*, and make *DF* equal to twice *DE*. Join *AE*, *FB*; and produce them to meet in *G*. Join *AF*; *AFG* is the triangle required.

Join *CE*. Since *BD* is double of *DC*, and *DF* double of *DE*; *EC* is parallel to *FG*; and ∴ the angle *AGF* is equal to *AEC*, *i. e.* to the given angle. Hence also *AE* is equal to *EG*; and *BF* being double of *EC*, is equal to *BG*; ∴ *AB* and *FE*, equal to the given lines, are drawn to the points of bisection of the sides.

* * *

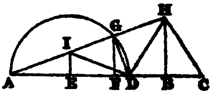
(35.) *Given the lengths of three lines drawn from the angles to the points of bisection of the opposite sides; to construct the triangle.*



Describe a triangle *ABC* whose sides are respectively equal to two thirds of the given lines; and complete the parallelogram *ABDC*. Join *AD*; and produce *CB* to *E*, making *BE* = *BC*. Join *AE*, *ED*; *AED* is the triangle required.

Produce *AB*, *DB* to *F* and *G*. Since the diagonals of parallelograms bisect each other, *AH* is equal to *HD*, and *BH* to *HC*; ∴ *EH* is equal to *EB* and *BH* together, *i. e.* to *BC* and *BH* or to $\frac{3}{2}$ *BC* and ∴ is equal to one of the given lines. Again, since *GB* is parallel to *AC*, (Eucl. vi. 2.) it bisects *AE*, and $BG = \frac{1}{2}AC$; but $DB = AC$, $\therefore DG = \frac{3}{2}AC$, and ∴ is equal to another of the given bisecting lines. In the same manner, *AF* may be shewn to be equal to the other given line, and to bisect *DE* in *F*.

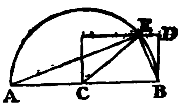
(36.) *Given the segments of the base made by the perpendicular, and one of the angles at the base triple the other; to construct the triangle.*



Let *AB*, *BC* equal to the given segments, be placed in the same straight line. Make $BD = BC$; bisect *AD* in *E*, and *BE* in *F*. On *AD* describe a semicircle; and from *F* draw *FG* at right angles to *AD*. Join *AG*, *GD*; and let *AG* meet the perpendicular *BH* in *H*. Join *HC*; *AHC* is the triangle required.

Draw *EI* perpendicular to *AD*; join *DI*, *DH*. Then *AE* being equal to *ED* and the angles at *E* right angles, $AI = ID$, and the angle $IAD = IDA$; whence the angle *DIH* is double of *DAH*. But since *EF* is equal to *FB*, and *GF* parallel to *IE* and *BH*, $\therefore IG = GH$; and the angles *DGI*, *DGH* being right angles, *DI* is equal to *DH*, and the angle *DHI* equal to *DIH*, and ∴ double of *DAH*. Also since $DB = BC$, and the angles at *B* are right angles, ∴ the angle *HCB* is equal to *HDB*, *i. e.* to *DHA*, *DAH* together, or to three times the angle *DAH*. Also *AB*, *BC*, by construction, are equal to the given segments made by the perpendicular.

(37.) *The area and hypothenuse of a right-angled triangle being given; to describe the triangle.*

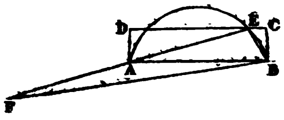


Let AB be equal to the given hypothenuse. Bisect it in C, and on CB describe a rectangular parallelogram CD equal to the given area. On AB describe a semicircle AEB, cutting the side parallel to AB, in E. Join AE, EB; AEB is the triangle required.

Join CE. Since $AC = CB$, the triangle AEB is double of CEB (Eucl. i. 38.), and ∴ equal to the rectangle CD, *i. e.* to the given area.

* * *

(38.) *Given one angle, and a line drawn from one of the others bisecting the side opposite to it; to construct the triangle, when the area is also given.*

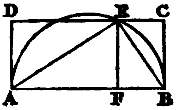


Let AB be the given bisecting line; and upon it describe a segment of a circle containing an angle equal to the given angle. Upon AB also describe a rectangular parallelogram $ABCD$ equal to the given area; and let DC meet the circle in E; join EA, and produce it, making $AF = AE$; join FB, BE; FEB is the triangle required.

For BA bisects the side EF; the angle BEF is equal to the given angle; and the triangle BEF is double of BAE (Eucl. i. 38.) and ∴ equal to $ABCD$, *i. e.* to the given area.

* * *

(39.) *In two similar right-angled triangles, the sum of the base of the one and perpendicular of the other* *is given; to determine the triangles such that their hypothenuses may contain the right angle of another triangle similar to them; and the sum of the three areas may be equal to a given area.*

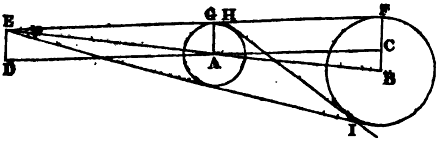


Let *AB* be equal to the given sum; and upon it describe a rectangular parallelogram equal to the given area. On *AD* also describe a semicircle, cutting *CD* in *E*; join *AE*, *EB*; the triangles *AED*, *BEC*, *AEB* are the triangles required; as is evident from the construction.

* * *

(40.) *Given the vertical angle, the area, and the distance between the centres of the inscribed circle and the circle which touches the base, and the two sides produced; to construct the triangle.*

Let *AB* be equal to the given distance between the centres; and make the angle *BAC* equal to half the given angle at the vertex. On *AC* let fall the perpendicular *BC*; and produce *CA*, till the rectangle *AD*, *DC* is to the given area, in the ratio of *AC* : *CB*. Complete the parallelogram *DEFC*; and from the centres *A* and *B*, describe two circles touching *EF* in *G* and *F*. Draw *HI*, *IE* touching the two circles; *HIE* is the triangle required.



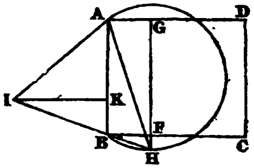
For AB is equal to the given distance between the centres. The angle IEF is double of AEF, *i. e.* of BAC, and $\therefore$ is equal to the given angle. And from the similar triangles ABC, AED,

$$AC : CB :: AD : (DE=)\, AG :: AD \times DC : AG \times EF.$$

But since EF is equal to half the perimeter of the triangle EHI, the rectangle AG, EF is equal to the triangle EHI; whence EHI is equal to the given area.

* * *

(41.) *Given the area, the line from the vertex dividing the base into segments which have a given ratio, and either of the angles at the base; to construct the triangle.*



On AB the given line, describe a segment of a circle containing an angle equal to the given angle. Describe a rectangular parallelogram $ABCD$ equal to twice the given area; and divide BC in the given ratio at F. Through F draw GH parallel to AB. Join BH, HA; and produce HB so that IB may be to BH in the given ratio. Join IA; IAH will be the triangle required.

Draw IK perpendicular to AB. The triangles IKB, BFH are similar,

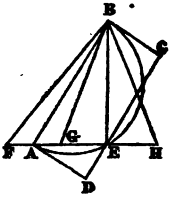
$$\therefore IK : BF :: IB : BH :: FC : FB,$$

$\therefore IK = FC$; and the triangle IAB is equal to half of the parallelogram GC; and the triangle ABH is equal to half of BG; $\therefore IAH$ is equal to half the parallelogram

AC, and ∴ equal to the given area. And *AB* is equal to the given dividing line, and *IB* : *BH* in the given ratio.

* * *

(42.) *Given the difference between the segments of the base made by the perpendicular, the sum of the squares of the sides, and the area; to construct the triangle.*

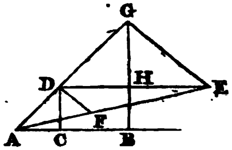


Take a line *AB* such that its square may be equal to the difference between half the given sum of the squares, and the square of half the given difference of the segments of the base. Upon *AB* describe a rectangular parallelogram *ABCD* equal to the given area; and also on *AB* describe a semicircle cutting *CD* in *E*. Join *AE*, and produce it both ways; and make *AF*, *AG*, each equal to half the given difference of the segments, and make *EH*=*EG*. Join *BF*, *BH*; *BFH* is the triangle required.

Join *BG*, *BE*. Since *GE*=*EH* and the angles at *E* are right angles, ∴ *GB*=*BH*; and the sum of the squares of *FB*, *BH* is equal to the sum of the squares of *FB*, *BG*, *i. e.* (iv. 30.) is equal to twice the sum of the squares of *FA* and *AB*, *i. e.* by construction, is equal to the given sum. Also the difference between *FE* and *EH* is equal to the difference between *FE* and *EG*, *i. e.* to *FG*, which is equal to the given difference. And the area of the triangle *FBH* is double of the area of the triangle *ABE*, and ∴ (Eucl. i. 41.) equal to *ABCD*, or the given area.

* * *

(43.) *Given the base, one of the angles at the base, and the difference between the side opposite to it and the perpendicular; to construct the triangle.*



Take an indefinite line *AB*; and from any point *C* in it draw *CD* perpendicular to it, and equal to the given difference. From *D* draw *DA* making the angle *DAC* equal to the given angle. Draw *DE* parallel to *AB*, and equal to the given base. Join *AE*; to which from *D* draw *DF*=*DC*. Draw *EG* parallel to *FD*, and meeting *AD* produced; *EDG* will be the triangle required.

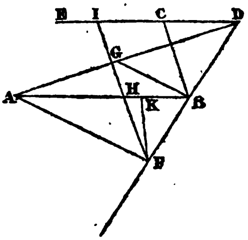
From *G* draw *GB* perpendicular to *AB*. Since *DF* is parallel to *GE*, and *DC* to *GB*,

DC : *GB* :: *AD* : *AG* :: *DF* : *GE*.

But *DC* being equal to *DF*, *GB* is equal to *GE*; whence the difference between *GE* and *GH* is equal to *HB*, or *DC*, *i. e.* to the given difference. And the angle *GDE* is equal to *DAC*, *i. e.* to the given angle; and *DE* is equal to the given base.

(44.) *Given the vertical angle, the difference of the base and one side, and the sum of the perpendicular drawn from the angle at the base contiguous to that side upon the opposite side and the segment cut off by it from that opposite side contiguous to the other angle at the base; to construct the triangle.*

Let *AB* be equal to the given sum, and *BC* to the given difference; and let them be placed so as to contain



an angle ABC, which with the given vertical angle will be equal to two right angles. Through C draw DCE parallel to AB, and through B draw DBF making half a right angle with it. Join AD; and to it from B draw BG equal to BC; and parallel to these respectively draw AF, FH; AFH will be the triangle required.

Produce FH to I; and let fall the perpendicular FK. Since BC is parallel to FI, and BG to AF,

$$BC : FI :: BD : FD :: BG : AF,$$

but $BC=BG$, $\therefore$ $FI=AF$; and $HI=BC$ will be the difference between the base AF and the side FH. Also since the angle BFK is half a right angle, $FK=KB$; and AB is the sum of the perpendicular FK and the segment KA. Also $BCIH$ being a parallelogram, the angle AHF, or its vertically opposite IHB, together with ABC will be equal to two right angles; and $\therefore$ AHF is equal to the given vertical angle.

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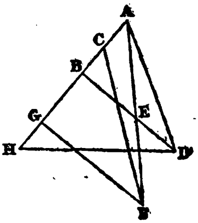
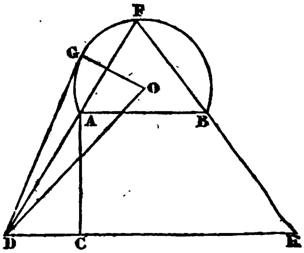
(45.) ***Given the base, the difference of the sides, and the segment intercepted between the vertex and a perpendicular from one of the angles at the base upon the opposite side; to construct the triangle.***



Let AB be equal to the given segment intercepted between the perpendicular and the vertical angle; BC equal to the given difference of the sides; draw BD perpendicular to AB, and make $BE=BA$; join AE, and produce it; and from C to AE draw CF such that its square may be equal to the given squares of the base and segment AB; draw FG perpendicular to AB; and make $GH=BC$. From H to BD, draw HD equal to the given base; join AD; AHD is the triangle required.

For AB is made equal to the given segment, and HD to the given base. Also since $AB=BE$, $AG=GF$, and $HB=GC$; whence the squares of GA and HB are equal to the squares of GA and GC, *i. e.* to the square of CF, or the squares of HD and AB, by construction. But (iv. 29.) the squares of AD and HB are equal to the squares of HD and AB; $\therefore$ the squares of GA and HB are equal to the squares of AD and HB; whence $GA=AD$; and $\therefore$ the difference between AH and AD is equal to HG, *i. e.* to BC, or the given difference.

(46.) *Given the vertical angle, the side of the inscribed square, and the rectangle contained by one side and its segment adjacent to the base made by the angular point of the inscribed square. To construct the triangle.*



Let AB be equal to a side of the inscribed square; and upon it describe a segment of a circle containing an angle equal to the given angle. From A draw AC perpendicular and equal to AB; and through C draw DCE parallel to AB. Find O the centre of the circle; and from it to DE, draw OD such that the difference of the squares of OD

and the radius of the circle may be equal to the given rectangle. Join *DA*, and produce it to *F*. Join *FB*, and produce it to *E*; *DFE* is the triangle required.

From *D* draw the tangent *DG*. Join *GO*. Then the squares of *DG*, *GO* are together equal to the square of *DO*, *i. e.* to the square of *GO* and the given rectangle; and ∴ the square of *DG*, *i. e.* the rectangle *AD*, *DF* is equal to the given rectangle; ∴ *DF* is a side of the triangle, in which *AB* is the side of an inscribed square. And *DFE* is equal to the given vertical angle.

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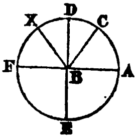
APPENDIX.

PLANE TRIGONOMETRY.

1. Def. 1. *Plane Trigonometry* is that branch of Mathematical science which treats of the measures of angles and the relation between the respective sides and angles of plane triangles.

Lemma I.

(2.) *If from the centre of a circle two straight lines be drawn to its circumference, the included angle will be to four right angles, as the intercepted arc is to the whole circumference.*

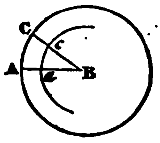


Let BA, BC be two straight lines drawn from the centre B of the circle ADE, meeting the circumference in A and C; the angle ABC will be to four right angles as the arc AC is to the whole circumference.

Produce AB till it meets the circumference again in F; and through B draw DE perpendicular to AB, meeting the circle in D and E. Then (Eucl. vi. 33.) the angle ABC : the right angle ABD :: the arc AC : AD; and quadrupling the consequents, the angle ABC : four right angles :: the arc AC : $4AD$, or the whole circumference.

LEMMA II.

(3.) *If from the common centre of two circles two straight lines be drawn intersecting the circumferences; the intercepted arc of either circle will be to the intercepted arc of the other circle, as the circumference of the first circle to the circumference of the second.*



From B the common centre of two circles are drawn the lines BA, BC, intersecting the circumferences in A, C, a, c; the arc AC : ac :: the whole circumference of the first circle : the whole circumference of the second.

For (2) the arc AC : the whole circumference (C) of which it is an arc :: the angle ABC : four right angles; which is the same ratio with that of ac : the whole circumference (C') of which it is an arc,

$$\therefore \text{(Eucl. v. 15.)} \quad AC : C :: ac : C',$$
$$\text{and } alt. \quad AC : ac :: C : C'.$$

(4.) COR. 1. If the circumferences of any two circles be severally divided into the same number of equal parts, whatever number of such parts is contained in any arc (AC) of one circle, the same number will be contained in that arc (AC) of the other circle which subtends the same angle.

(5.) COR. 2. Any arc of a circle is the measure of the angle which it subtends at the centre.

For (See Fig. 1.) draw any straight line Bx to the circumference; then (Eucl. vi. 33.) the arc AC : the arc Ax :: the angle ABC : the angle ABx, *i. e.* in whatever proportion the angle at the centre is increased, the subtending arc is increased in the same proportion *.

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* If the angle be not at the centre, but between it and the circumference;

(6.) Angles at the centres of different circles vary as the arcs which subtend them, directly, and the radii of the circles inversely.

For (2) the arc AC : the circumference $ADEA$ :: the angle ABC : four right angles. Hence the angle $ABC = \frac{\text{the arc } AC}{ADEA} \times$ four right angles; and since the circumference $ADEA \propto AB$, and four right angles is an invariable quantity, the angle $ABC \propto \frac{AC}{AB}$.

(7.) Till lately, Geometers *almost unanimously* agreed to divide the circumference of the circle into 360 equal parts called Degrees; each degree into 60 equal parts called Minutes; and each minute into 60 equal parts called Seconds, &c. This method appeared to the Greek Geometers to afford some facilities for calculations, in consequence of the great number of divisors of 360 and 60; but it is in reality subject to inconvenience from complicated numbers, and tediousness in operations. Upon the introduction therefore of a new system of weights and measures into France, in which they were decimally divided and subdivided, it was thought proper to make a similar division of the quadrant. The quadrant is accordingly divided into 100 minutes, each minute into 100 seconds, &c. One great advantage of which method is its identity with the common decimal scale of

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ference; it will be measured by half the sum of the arcs intercepted between the sides, and the sides produced. If it be without the circle, it will be measured by half their difference (ii. 24.).

If the angle be formed by a tangent and chord, it will be measured by half the arc subtended by the chord. If by a line cutting the circle and a tangent, or by two tangents, it will be measured by half the difference between the convex and concave arcs intercepted between the lines.

notation; thus 6 deg. 15 min. 53 sec. would be represented by 6.1553 deg. and 34 deg. 5 min. 9 sec. by 34.0509.

The length of a degree then on that scale will be *less* than one on the common scale in the proportion of 90 : 100, or 9 : 10; a minute in the proportion of 90 × 60 : 100 × 100, or 27 : 50, and a second in the proportion of 90 × 60 × 60 : 100 × 100 × 100, or 81 : 250.

If then n = the number of degrees on the new scale, the corresponding number on the former scale will be $= \frac{n \times 9}{10} = \frac{n.(10-1)}{10} = n - \frac{n}{10}$. Hence ∴ the following rule, for reducing the new to the old graduation.

Express the measure in decimals, and from it subtract $\frac{1}{10}$th of this number; mark off the decimals in the remainder, which multiply by 60, and mark off the decimals again; these again multiply by 60 and mark off as before, and so on; the whole numbers so obtained will be the degrees, minutes, seconds, &c. on the common scale.

Thus to determine how many degrees, minutes, &c. in the old graduation, correspond to 46 deg. 43 min. 15 sec. in the new.

From 46.4315
Subtract 4.64315
41.78835 *
60
47.30100
60
18.06000

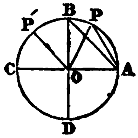
or 41 deg. 47 min. 18.06 sec.

* To find the new from the old add $\frac{1}{9}$ of old to itself, or × by 10 and ÷ by

(8.) Degrees, Minutes, Seconds, &c. are expressed by °, ′, ″; thus 10°. 15′. 45″. represent an arc of ten degrees, fifteen minutes and forty five seconds.

(9.) Arcs and angles are expressed without distinction in trigonometrical calculations by degrees, minutes, and seconds.

(10.) A right angle is measured by an arc of 90°; two right angles by 180°.; half a right angle by 45°; and each angle of an equilateral triangle by 60°.



(11.) Def. 2. With the centre *O* and radius *OA* describe a circle, and draw the diameters *AOC*, *BOD* at right angles to each other. These will divide the circumference into four equal parts called *Quadrants*.

(12.) Def. 3. The *Complement* of any arc or angle is the difference between that arc or angle and a quadrant or 90°. Thus taking any point *P* in the quadrant *AB*; the complement of the arc *AP* is the arc *BP*; and the complement of the angle *AOP* is the angle *BOP*. If $AP = 62°. 16'$, *BP* its complement $= 27°. 44'$. Also the complement of 35°. 15′. 45″ is an arc of 54°. 44′. 15″. And in general *A* being any arc whatever, its complement is $90° - A$; or $A - 90°$, if *A* be greater than 90°.

The two acute angles of a right-angled triangle, being together equal to a right angle, are complements to each other.

(13.) Def. 4. The *Supplement* of any arc or angle is its defect from a semicircle or 180°. Thus, taking any point *P*, the supplement of the arc *AP* is *PBC*, and of the angle *AOP* is the angle *POC*; the supplement of *ABP′* is *CP′*, and of the angle *AOP′* is the angle *COP′*. The supplement of 32°. 42′. 18″ is an arc of 147°. 17′. 42″.

If A be any arc, its supplement is $(180-A)$. Also the supplement of $(90-A)$ is $(90+A)$.

In every triangle, the sum of the three angles being equal to two right angles, each angle is the supplement to the sum of the other two.

(14.) Def. 5. The *Chord* of an arc is a straight line joining the extremities of the arc. Thus AP is the chord of the arc AP. AB is the chord of the quadrant AB. The diameter is the chord of a semicircle; and the chord of the whole circumference is $=0$.

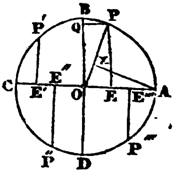
(15.) If the chord does not pass through the centre, as AP, it is the chord of two unequal arcs, AP and the remaining part of the circumference $ADCBP$. In speaking however of the chord of an arc of a circle, the less arc is generally meant.

(16.) Cor. The chord of $60° = $ radius of the circle. For the triangle AOP is equiangular and $\therefore$ equilateral.

The chord of a quadrant $= R\sqrt{2}$.

(17.) Def. 6. The *Sine* or *Right Sine* of an arc is a straight line drawn from one end of the arc perpendicular to the diameter passing through the other end of the arc.

Thus, if from P, PE be drawn perpendicular to AC, it will be the sine of the arc AP less than a quadrant. If PQ be perpendicular to BO, it will be the sine of the arc BP. Also $P'E'$, $P''E''$, $P'''E'''$ are the sines of arcs AP', AP'', AP''' measured from A and terminating in the second, third and fourth quadrants respectively.



If from A, Ax be drawn perpendicular to OP, it will be the sine of AP, and be equal to PE.

(18.) To trace the changes in magnitude of the Right Sine.

The sine begins with the arc; when the arc is nothing, P coinciding with A, E also coincides with it. But as P moves from A towards B, PE continually increases, *i. e.* as the arc increases, the sine increases; and when P coincides with B, E coincides with O, and the sine of 90° becomes radius. When the arc is greater than 90°, and terminates in the second quadrant, as the arc increases, the sine decreases, till it becomes $= 0$, at the end of the second quadrant, P' and E' both coinciding with C. If the arc terminates in the third quadrant, the sine continually increases, till at the end of it, it becomes radius; and if in the fourth quadrant, it continually decreases again till it becomes $= 0$.

Hence it appears that the sine never exceeds radius.

(19.) To trace the changes in the Algebraic Sign of the right sine.

If the right sines of arcs in the first quadrant be *reckoned* positive, they will be positive or negative for arcs which terminate in the other quadrants, according as they are drawn in the same or contrary directions with the former, *i. e.* according as they fall on the same or contrary sides of the diameter AC. Hence during the two first quadrants the algebraic sign is positive, and for arcs which terminate in the two next it is negative.

(20.) Cor. The sine of an arc is equal to the sine of its supplement.

For PE is equal to the sine of CBP which is the supplement of AP. Thus 48°. 15′ being the supplement of 131°. 45′, the sine of 131°. 45′ = the sine of 41°. 15. Also $\sin. (180 - A) = \sin. A$, and $\sin. (90 + A) = \sin. (90 - A)$.

The exterior angle of a triangle being the supplement of its interior adjacent angle, their sines are equal;

(21.) Def. 7. The *Versed Sine* is that part of the diameter passing through the beginning of the arc, which is intercepted between the beginning of the arc and the right sine.

Thus *AE* is the versed sine of the arc *AP*. *AE'*, *AE''*, *AE'''* are the versed sines of the arcs *AP'*, *AP''*, *AP'''* respectively.

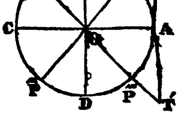
(22.) To trace the changes in magnitude of the versed sine.

The versed sine begins with the arc, and when the arc is nothing, the versed sine $=0$, *E* coinciding with *A*. It increases with the arc, till the arc becoming a semicircle, the versed sine becomes the diameter; after which it again decreases, whilst the arcs terminate in the third and fourth quadrants, till at the end of the fourth quadrant it becomes $=0$. But as it is always measured in the same direction from *A*, it is positive through all the four quadrants.

(23.) Cor. Hence (Eucl. vi. 8.) the chord of an arc is a mean proportional between the diameter and the versed sine of that arc.

(24.) Def. 8. The *Tangent* of an arc is a straight line which touches the circle at one end of the arc, and is terminated by the radius produced through the other end of the arc.

Thus, if the line *TAT'* touch the circle in *A*, and *AP* be an arc less than a quadrant, and *OP* joined; since *TAO*, *AOP* are less than two right angles, *OP* and *AT* will meet above *AO*; and *AT*



will be the tangent of *AP*. But if *AP'* be taken an arc greater than a quadrant, and less than a semicircle, *TAO*, *AOP'* are greater than two right angles, and ∴ *TA* and *P'O* will meet on the contrary side of *AO*, or *AT'* will be the tangent of *ABP'*. In the same manner it may be shewn that *AT* is the tangent of an arc *ABCP''* terminating in the third quadrant; and *AT''* of one terminating in the fourth.

(25.) To trace the changes in magnitude of the tangent.

The tangent begins with the arc; when the arc is nothing, *P* coinciding with *A*, *T* coincides with it also. Whilst the arc is less than a quadrant, as *AP* increases, the angle *AOT*, and ∴ *AT* increases, or the tangent increases; and as the arc approximates to a quadrant, *OP* approximates to the direction *OB* which is parallel to *AT*, and ∴ the tangent approximates to a line greater than any that can be assigned. But if the arc be greater than a quadrant, and less than a semicircle, as the arc increases, *BP'* increases, ∴ the angle *BOP'* or *DOT'* increases, and *AOT'* decreases, ∴ *PT'* intersects *AT* in points continually nearer and nearer to *A*, or the tangent decreases, and at the end of the second quadrant, the tangent = 0. In the same manner it may be shewn that the tangent of an arc terminating in the third quadrant continually increases till it approximates to a line greater than any that can be assigned; and for arcs terminating in the fourth quadrant, it decreases till it be becomes = 0.

(26.) To trace the changes in the algebraic sign of the tangent.

If the tangent of an arc in the first quadrant be *reckoned* positive, it will be positive or negative for arcs terminating in the other quadrants, according as it is drawn in the same or contrary directions with the tangents of arcs ending in the first quadrant, *i. e.* according as it falls on the same or opposite sides of the diameter *AC*. Hence the tangents of arcs ending in the first and third quadrants have a positive sign, and those of arcs ending in the second and fourth, a negative one.

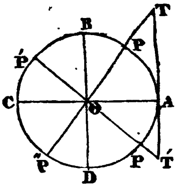
27. Cor. 1. The tangent of an arc is equal to the tangent of its supplement;

For the angle *POC* is equal to the vertically opposite angle *AOP″*; and (24) *AT* is the tangent of *ADP″* which measures *AOP″*. As one of the arcs is less, and the other greater than a quadrant, they will be affected with contrary signs.

Hence if A be any angle, tang. $(180-A) = -$tang. A and tang. $(90+A) = -$tang. $(90-A)$.

(28.) Cor. 2. Tang. $45° =$ radius. For in the triangle *OAT*, the angle *OAT* is a right angle and $AOT = 45°$, $\therefore ATO = 45° = AOT$ and $AT = AO$.

29. Def. 9. The *Secant* of an arc is the line drawn from the centre through the end of the arc, and produced till it meets the tangent;



Thus, the construction remaining as in Art. 24. *OT* is the secant of the arc *AP*; and the secant of the arc *AP′* terminating in the second quadrant is *OT′*. The secants also of arcs *AP″*, *AP‴* terminating in the third and fourth quadrants will be *OT*, *OT′* respectively.

(30.) To trace the changes in magnitude of the secant.

When P coincides with A, the secant coincides with the radius *OA*, *i.e.* in the beginning of the circle the secant = radius. In the first quadrant as *AP* increases, *AT* increases (25.) ∴ *OT* increases (Eucl. i. 47.) *i.e.* as the arc increases, the secant increases; and when the arc approximates to a quadrant, the secant approximates to a line greater than any that can be assigned. But if the arc be greater than a quadrant and less than a semicircle, as the arc increases, *T′* approaches nearer to *A*, and the secant decreases, till at the end of the second quadrant, it becomes equal to radius. In the same manner, the secant of an arc ending in the third quadrant increases from radius till it approximates to a line greater than any that can be assigned; and for arcs terminating in the fourth quadrant, decreases again till it becomes = radius.

Hence the secant is never less than the radius.

(31.) To trace the changes in the algebraic sign of the secant.

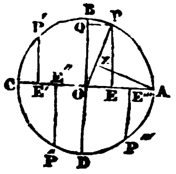
If the secants of arcs in the first quadrant are *reckoned* positive; those of arcs ending in the other quadrants will be positive or negative according as they are drawn from the centre through the end of the arc, or from the end of the arc through the centre. Hence the secants of arcs ending in the first and fourth quadrants have a positive sign; and of those ending in the second and third, a negative sign.

(32.) Cor. 1. The secant of an arc is equal to the secant of its supplement.

For *OT* is the secant of *ADP″* which is the supplement of *AP*. As one of the arcs is greater and the other less than a quadrant, the secants will have contrary signs. If A be any angle, sec. $(180-A) = -$sec. A, and sec. $(90-A) = -$sec. $(90+A)$.

(33.) Cor. 2. Sec. $45° = R\sqrt{2}$.

(34.) Def. 10. The *Cosine* of an arc is that part of the diameter passing through the beginning of the arc, which is intercepted between the centre and the right sine.



Thus OE is the cosine of the arc AP in the first quadrant; OQ the cosine of the arc BP. Also OE', OE'', OE''' are the cosines of arcs AP', AP'', AP''' respectively terminating in the second, third and fourth quadrants.

(35.) To trace the changes in magnitude of the cosine.

When P coincides with A, E also coincides with it, and in the beginning of the arc, cosine$=$radius. In the first quadrant, as AP increases, EO decreases, *i. e.* as the arc increases, the cosine decreases; till at the end of the first quadrant, the cosine$=0$. But if the arc be greater than a quadrant, and less than a semicircle, as AP' increases, OE' increases, and $\therefore$ as the arc increases the cosine increases, till at the end of the second quadrant it $=$radius. In the same manner the cosines of arcs terminating in the third quadrant decrease from radius, till at the end of the third quadrant the cosine$=0$, and in the fourth quadrant increase till at the end of it, cosine $=$radius.

Hence the cosine is never greater than radius.

(36.) To trace the changes in the algebraic sign of the cosine.

If the cosines of arcs ending in the first quadrant be *reckoned* positive, those of arcs terminating in the other quadrants will be positive or negative according as they are drawn in the same or opposite directions with those of arcs ending in the first quadrant, *i.e.* according as they are measured on the same or opposite sides of the centre with the cosines of arcs in the first quadrant. Hence the cosine has a positive sign in the first and fourth quadrants, and a negative one in the second and third.

(37.) Cor. 1. The cosine of an arc is equal to the cosine of its supplement.

For *OE* is the cosine of the arc *CBP* which is the supplement of *AP*. But as one of the arcs is less and the other greater than a quadrant, they will be affected with contrary signs. Hence if *A* be any arc, $\cos.(180-A) = -\cos. A$; and $\cos. (90+A) = -\cos. (90-A)$.

(38.) Cor. 2. The cosine of an arc is equal to the sine of its complement.

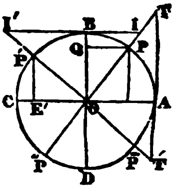
For *OE* the cosine of *AP* is equal to *PQ* the sine of *BP*. Hence $\cos. A = \sin. (90-A)$ and $\cos. (90-A) = \sin. A$. If $A = 30°$, $\cos. 30° = \sin. 60°$, and $\cos. 60° = \sin. 30°$.

(39.) Cor. 3. $\text{Cos.} 45° = \sin. 45° = \dfrac{R}{\sqrt{2}}$. (Eucl. i. 47.)

(40.) Cor. 4. The versed sine of an arc less than a quadrant is equal to the difference of the radius and cosine; and of an arc greater than a quadrant is equal to their sum.

(41.) Def. 11. The *Cotangent* of an arc is a line touching the circle at the end of the first quadrant and meeting the radius produced through the end of the arc.

Thus, if *IBI′* be drawn touching the circle in *B* the end of the quadrant *AB*, *BI* will be the cotangent of the arc *AP* which is less than a quadrant. But if the arc



arc ABP' be greater than a quadrant and less than a semicircle, BI' is its cotangent. Also the cotangents of the arcs AP'', AP''' ending in the third and fourth quadrants are respectively BI and BI'.

(42.) To trace the changes in magnitude of the cotangent.

When P coincides with A, OPI falls in the direction OA, and being parallel to BI, the cotangent is greater than any assignable line. In the first quadrant, as the arc AP increases, OI intersects the line BI in points continually nearer to B. Hence in arcs less than a quadrant, as the arc increases, the cotangent decreases, till at the end of the first quadrant it$=0$. In arcs greater than a quadrant and less than a semicircle, as the arc ABP' increases, BI' the cotangent increases; till as the arc approximates to a semicircle, the cotangent approximates to a line greater than any that can be assigned. In the same manner the cotangents of arcs ending in the third quadrant continually decrease, till at the end of it, the cotangent$=0$; and if the arcs terminate in the fourth quadrant, the cotangent increases again, till it approximates to a line greater than any that can be assigned.

(43.) To trace the changes of the cotangent in the Algebraic Sign.

If the cotangents of arcs ending in the first quadrant be *reckoned* positive, those of arcs ending in the other quadrants will be positive or negative according as they are drawn in the same or contrary directions with those of arcs ending in the first quadrant, *i. e.* according as they fall on the same side of the diameter BD with the cotangents of arcs ending in the first quadrant or the contrary.

Hence the cotangents of arcs ending in the first and third quadrants have a positive, those of arcs ending in the second and fourth, a negative sign.

(44.) Cor. 1. The cotangent of an arc is equal to the cotangent of its supplement.

For BI is the cotangent of CBP which is the supplement of AP. But these cotangents have contrary signs. If $\therefore A$ be any arc, cot. $(180-A) = -\text{cot.}\ A$, and cot. $(90+A) = -\text{cot.}\ (90-A)$.

(45.) Cor. 2. The cotangent of an arc is equal to the tangent of its complement.

For BI the cotangent of AP is the tangent of BP which is the complement of AP. If $\therefore A$ be any arc, cot. $A = \text{tang.}\ (90-A)$ and cot. $(90-A) = \text{tang.}\ A$.

(46.) Cor. 3. Cot. $45 = \text{radius}$.

(47.) Def. 12. The *Cosecant* of an arc is a straight line drawn from the centre through the end of the arc and produced till it meets the cotangent.

Thus the arc AP being less than a quadrant, OI is its cosecant. Also OI' is the cosecant of the arc ABP' greater than a quadrant and less than a semicircle. And OI, OI' are the cosecants respectively of the arcs AP'', AP''' terminating in the third and fourth quadrants.

(48.) To trace the changes in magnitude of the cosecant.

When P coincides with A, OPI falls in the direction of OA which is parallel to BI, and therefore at the beginning of the arc the cosecant is greater than any assignable line. In the first quadrant, as the arc AP increases, the line OI cuts BI in points continually nearer to *B*, *i. e.* as the arc increases, the cosecant descreases, till at the end of the first quadrant it becomes equal to radius. As the arc increases from a quadrant to a semicircle, *I'B* increases (42) and the cosecant increases, till as the arc approximates to a semicircle, the cosecant approximates to a line greater than any that can be assigned. If the arc terminates in the third quadrant, as the arc increases, the cosecant decreases, till it becomes equal to radius; and during the fourth quadrant it increases again, till it approximates to a line greater than any that can be assigned.

Hence the cosecant is never less than radius.

(49.) To trace the changes in the algebraic sign of the cosecant.

The cosecants of arcs terminating in the first quadrant being *reckoned* positive, and drawn from the centre through the ends of the arcs; the cosecants of arcs ending in the other quadrants will be positive or negative according as they are drawn from the centre through the ends of the arcs, or from the ends of the arcs through the centre. And hence the cosecants of arcs ending in the two first quadrants have a positive sign, and those of arcs ending in the two last a negative sign.

(50.) Cor. 1. The cosecant of an arc is equal to the cosecant of its supplement, and they have the same sign. If ∴ A be any arc, cosec. $(180-A)$ = cosec. A, and cosec. $(90+A)$ = cosec. $(90-A)$.

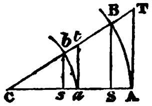
(51.) Cor. 2. The cosecant of an arc is equal to the secant of its complement; for *OI* the cosecant of the arc *AP* is the secant of *BP* the complement of *AP*. Hence cosec. A = sec. $(90-A)$ and cosec. $(90-A)$ = sec. A.

(52.) Cosec. $45° = R\sqrt{2}$. (33.)

(53.) In trigonometrical calculations, when the quantity required is deduced in terms of the sine, the case is ambiguous, since the sine of any arc and its supplement (19) have the same algebraic sign. The ambiguity may however be removed by some other consideration. But if an expression be deduced in terms of a cosine or a tangent, there will be no ambiguity, since a positive cosine (36) or tangent (26) denotes an arc less than 90°, and a negative cosine or tangent indicates an arc greater than 90° and less than 180°. In calculations arcs and angles are generally less than 180°.

Prop. I.

(54.) *The sines, cosines, tangents, &c. of the same angles at the centres of different circles are proportional to the radii of those circles.*



Let AB, ab be two arcs described with the centre C and radii CA, Ca, subtending the same angle at C. Draw BS the sine and AT the tangent of AB; bs the sine and at the tangent of ab. Then the triangles CBS, Cbs being similar,

$$BS : bs :: CB : Cb,$$

i. e. the sines are proportional to the radii.

Also $CS : Cs :: CB : Cb$,

or the cosines are proportional to the radii.

And since $CS : Cs :: CB : Cb :: CA : Ca$;

$\therefore$ (Eucl. v. 19.) $AS : as :: CA : Ca$,

or the versed sines are proportional to the radii.

Again from the similar triangles ATC, atC,

$$AT : at :: AC : aC,$$

or the tangents are proportional to the radii.

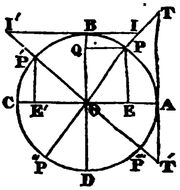
and $CT : Ct :: AC : aC$;
or the secants are proportional to the radii.

(55.) Cor. 1. Hence $\frac{BS}{BC} = \frac{bs}{bc}$ and ∴ if S and s represent the sines of the same angle to radii R and r, $\frac{S}{R} = \frac{s}{r}$, and $S = \frac{R}{r} \times s$. If $r = 1$, $S = Rs$, and $\frac{S}{R} = s$. And the same is true of any other corresponding lines. Hence if any expressions be calculated for rad. = 1; the corresponding expressions in a circle whose radius is R will be determined by taking each line an R^{th} part of the former.

(56.) Cor. 2. If a given radius be divided into any number of equal parts, and the sines &c. of every angle be given in such parts, the sines &c. of any given angle may be found, corresponding to another given radius.

Prop. II.

(57.) *To find the relation between the sines, cosines, tangents &c. of the same angle.*



Let AP be any arc, PE and OE its sine and cosine, draw the tangent AT, secant OT and cotangent BI. Then PE being parallel to AT, the triangles OPE, OAT are equiangular,

$$\therefore OE : EP :: OA : AT,$$

$$\text{or cos. : sine :: rad. : tang.}$$

$$\therefore \text{tang.} = \text{rad.} \times \frac{\text{sine}}{\text{cos.}} = \frac{\text{sine}^*}{\text{cos.}}, \text{ if } R = 1,$$

---

* In the second quadrant the sine is positive and the cosine negative. Hence

Also from the same triangles, $OE : OP :: OA : OT$, or $\cos : R :: R : \text{sec.}\ \therefore \text{sec.} = \frac{R^2}{\cos.} = \frac{1\text{[*]}}{\cos.}$ if $R=1$.

Also from the similar triangles OPE, BOI,
$PE : PO :: OB : OI$, or $\sin. : R :: R : \text{cosec.}$;

$$\therefore \text{cosec.} = \frac{R^2}{\text{sine}} = \frac{1\text{[†]}}{\text{sine}} \text{ (if } R=1).$$

And from the same triangles, $PE : EO :: BO : BI$,
or $\sin. : \cos. :: R : \text{cotang.}$;

$$\therefore \text{cotang.} = R \times \frac{\cos.}{\text{sine}} = \frac{\cos.\text{[‡]}}{\text{sine}}, \text{ if } R = 1.$$

Also from the similar triangles OAT, OBI,
$TA : AO :: OB : BI$, or $\text{tang.} : R :: R : \text{cotang.}$;

$$\therefore \text{cotang.} = \frac{R^2}{\text{tang.}} = \frac{1}{\text{tang.}}, \text{ if } R = 1.$$

(58.) COR. 1. If either the sine or cosine of an arc A be known, all the rest may be found.

Hence $\text{tang.}(180-A) = \frac{\sin. A}{-\cos. A} = -\frac{\sin. A}{\cos. A} = -\text{tang. } A$, which confirms what was shewn in Art 26.

When the sine and cosine have the same algebraic sign, the tangent will have a positive sign, and when different, a negative one. Also since $\cos. 90^\circ = 0$ and $\sin. 90^\circ = R$, $\text{tang. } 90^\circ = \frac{1}{0} = \infty$, which confirms what was shewn in Art. 25.

[*] Hence the secant will have the same algebraic sign as the cosine.

[†] Hence the cosecant will have the same algebraic sign as the right sine.

[‡] Hence the cotangent will have a positive sign, when the cosine and sine have the same algebraic signs, and a negative one, when different; or as also appears from the next equation, it will have the same sign as the tangent.

For (Eucl. i. 47.) $PO^2 = PE^2 + EO^2$, *i.e.* $R^2 = \sin^2. A + \cos^2. A$, ∴ $\sin. A = \sqrt{(R^2 - \cos^2. A)}$, and $\cos. A = \sqrt{(R^2 - \sin^2. A)}$, which values may be substituted in the preceding expressions.

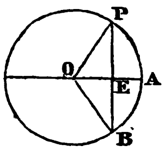
(59.) Cor. 2. Since $OT^2 = OA^2 + AT^2$, $\sec^2. A = R^2 + \text{tang}^2. A$, and ∴ $\sec. A = \sqrt{(R^2 + \text{tang}^2. A)}$, whence $\cos. A = \dfrac{R^2}{\sqrt{(R^2 + \text{tang}^2. A)}}$, and $\sin. A = \dfrac{R \times \text{tang}. A}{\sqrt{(R^2 + \text{tang}^2. A)}}$.

(60.) Cor. 3. Also $\text{tang}. A = \sqrt{(\sec^2. A - R^2)}$, and $\text{cosec}^2. A = R^2 + \text{cotang}^2. A$.

(61.) Cor. 4. If the radius be represented by unity, the expressions become, $\sin. A = \sqrt{(1 - \cos^2. A)}$, $\cos. A = \sqrt{(1 - \sin^2. A)}$, $\sec. A = \sqrt{(1 + \text{tang}^2. A)}$, $\text{tang}. A = \sqrt{(\sec^2. A - 1)}$, $\cos. A = \dfrac{1}{\sqrt{(1 + \text{tang}^2. A)}}$ and $\sin. A = \dfrac{\text{tang}. A}{\sqrt{(1 + \text{tang}^2. A)}}$, $\text{tang}. A \times \text{cotang}. A = 1$.

Prop. III.

(62.) *The sine of any arc is equal to half the chord of double the arc.*



Let the arc *PB* be double of *PA*. Join *OA*, *PB* intersecting each other in *E*. Since *PB* is double of *PA*, *PA* = *AB*, and ∴ the angles *POE*, *BOE* are equal, but *OPE* is equal to *OBE*, and *OE* common to the triangles *OPE*, *OBE*, ∴ (Eucl. i. 26.) *PE* is equal to *EB*, and ∴ is half of *PB*; and the angles *OEP*, *OEB* are equal, *i. e.* each is a right angle;

hence PE is the sine of the arc AP, and is half the chord of double the arc.

(63.) Cor. 1. Hence the sin. $30° = \frac{1}{2}$ radius. For (16) the chord of $60° =$ radius.

$$\text{Also cos. } 30° = \sqrt{(R^2 - \sin^2.30)} = \sqrt{\left(R^2 - \frac{R^2}{4}\right)} = \frac{R\sqrt{3}}{2};$$

$$\text{tang. } 30° = \frac{R}{\sqrt{3}}, \text{ and sec. } 30° = \frac{2R}{\sqrt{3}}.$$

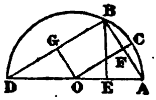
Hence sec. 30° is double the tang. 30°.

(64.) Cor. 2. $\text{Sin.} 60° = \frac{R\sqrt{3}}{2}$ (38), and $\cos. 60° = \frac{R}{2}$.

Versed sine $60° = \frac{R}{2}$, tang. $60° = R\sqrt{3}$, and sec. $60° = 2R$.

Prop. IV.

(65.) *The diameter is to the versed sine of any arc as the square of radius is to the square of the sine of half that arc.*



Let ABD be a semicircle, AD its diameter, ACB any arc, whose chord is AB and versed sine AE. Join DB, and from the centre O draw OFC perpendicular to the chord AB, and meeting the circumference in C; then (as was shewn above) AB is bisected in F, and the arc AB in C; and AF is the sine of $\frac{1}{2} AB$. Also the triangles DAB, OAF are similar,

$$\text{whence } DA : AB :: OA : AF.$$

But (Eucl. vi. 8. and v. Def. 10.)

$$DA : AE :: DA^2 : AB^2,$$

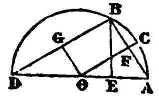
$$\therefore DA : AE :: OA^2 : AF^2.$$

(66.) **Cor. 1.** Hence the rectangle contained by the radius and the versed sine of any arc is equal to twice the square of the sine of half that arc.

(67.) **Cor. 2.** And the square of the sine of any arc varies as the versed sine of double that arc.

Prop. V.

(68.) *Radius is to the cosine of any arc as twice the sine of that arc is to the sine of double that arc.*



The same construction remaining, OF is the cosine of AC; and the triangles OAF, BAE, having the angle at A common, and right angles at F and E, are equiangular;

$$\text{whence } OA : OF :: (BA =)\, 2AF : BE.$$

(69.) **Cor. 1.** Hence the rectangle contained by the sine and cosine of any arc $\propto$ the sine of double that arc; for $BE \propto 2AF \times OF \propto AF \times OF$, *i.e.* $\sin.\ 2A \propto \sin.\ A \times \cos.\ A$.

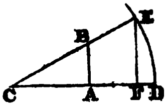
$$\text{Also } R \times \sin.\ A = 2 \sin.\ \tfrac{1}{2}A \times \cos.\ \tfrac{1}{2}A.$$

(70.) **Cor. 2.** If from O, OG be drawn perpendicular to DB, it bisects it, $\therefore DB = 2BG = 2OF = 2\cos.\ \tfrac{1}{2}AB$; or the chord of an arc is equal to twice the cosine of half the supplemental arc.

Prop. VI.

(71.) *In any right-angled triangle, radius is to the sine of either of the acute angles, as the hypothenuse is to the side opposite to that angle.*

Let ABC be a right-angled triangle, having the angle at



A a right angle. With the centre *C*, and radius *CD* = the tabular radius, describe an arc *DE*, meeting *CB* produced in *E*; it will be a measure of the angle *C*. Let fall the perpendicular *EF*, which will be equal to the tabular sine of the angle *C*. The triangles *CEF*, *CBA* being similar,

$$CE : EF :: CB : BA,$$

$$\text{or rad.} : \text{sin. } C :: CB : BA.$$

In the same manner, if *BA* be produced till it be equal to the tabular radius, and a circular arc be described with *B* as a centre, it may be shewn that rad. : sin. *B* :: *BC* : *CA*.

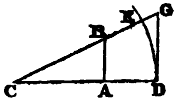
(72.) Cor. 1. Rad. : cos. *C* :: *CB* : *CA*;
and rad. : cos. *B* :: *CB* : *BA*.

(73.) Cor. 2. If rad. $= 1$, $BA = CB \times \text{sin. } C$, or $CB \times \text{cos. } B$,

and $CA = BC \times \text{sin. } B$, or $BC \times \text{cos. } C$.

Prop. VII.

(74.) *In any right-angled triangle, radius is to the tangent of either of the acute angles as the side adjacent to that angle is to the opposite side.*



Let *CAB* be a right-angled triangle, having the angle at *A* a right angle. With the centre *C*, and radius *CD* = the tabular radius, describe a circular arc *DE* meeting *CB* produced in *E*; then *DE* is the measure of the angle *C*; and drawing *DG* perpendicular to *DC* meeting *CE* in *G*, *DG* will be the tangent of *DE* or of the angle *ACB*. Now the triangles *DGC*, *ABC* being similar,

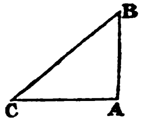
$$DC : DG :: AC : AB,$$
$$\text{or rad.} : \text{tang. } C :: AC : AB,$$

In the same manner it may be shewn that rad. : tang. B :: $AB : AC$.

(75.) COR. Rad. : sec. C :: $AC : BC$,
and rad. : sec. B :: $AB : BC$.

PROP. VIII.

(76.) *Having given two parts of a right-angled triangle; to find the other parts.*



This resolves itself into four cases.

1. Having given the hypothenuse CB, and the angle C; to find the rest.

Since (71) tabular radius : sin. C :: $BC : BA$, and the angle C being given, its sine may be found from the tables, the three first terms in the foregoing proportion are known, $\therefore BA = \frac{BC \times \text{sin. } C}{R}$ may be determined.

Also $AC = \sqrt{(BC^2 - BA^2)} = \sqrt{\{(BC+BA).(BC-BA)\}}$ may be computed, or three terms in the proportion (72) rad. : cos. C :: $BC : CA$ being known, $CA = \frac{BC \times \text{cos. } C}{R}$ may be also determined.

And the angle $B = 90° - C$.

Ex. Suppose the angle $C = 30°$ and $BC = 10$ chains. Then sin. $30° = \frac{1}{2} R$, and cos. $30° = \frac{\sqrt{3}}{2} R$, $\therefore BA =$

$\frac{10 \times \frac{1}{2} R}{R} = 5$, and $CA = \sqrt{(15 \times 5)} = 5\sqrt{3}$, or $CA = \frac{10 \times \sqrt{3} . R}{2 . R} = 5\sqrt{3}$, and $B = 90^\circ - 30^\circ = 60^\circ$.

By the same method the triangle may be solved, if the hypothenuse BC and the angle B be given; the point B being now considered as the centre.

2. Having given one side AC and the angle C; to find the rest.

Since (74) tabular radius : tang. C :: AC : AB, and the angle C being given, its tangent may be found from the tables; the three first terms in the preceding proportion are known, and $\therefore BA = \frac{AC \times \text{tang. } C}{R}$ may be determined.

Also (75) tabular radius : sec. C :: AC : CB, in which proportion three terms being known, the fourth, $CB = \frac{AC \times \text{sec. } C}{R}$ may be determined.

Or if in the tables the secants are not computed, tab. cos. C : R :: AC : CB, $\therefore CB = \frac{AC \times R}{\text{cos. } C}$, which may be determined.

Also the angle $B = 90^\circ - C$.

Ex. If $AC = 6$ chains, and the angle $C = 60^\circ$;

$$\text{tang. } 60^\circ = \sqrt{3} . R, \therefore AB = \frac{6 \times \sqrt{3} . R}{R} = 6\sqrt{3}.$$

$$\text{Also sec. } 60^\circ = 2R, \therefore CB = \frac{6 \times 2R}{R} = 12,$$

$$\text{or cos. } 60^\circ = \tfrac{1}{2} R, \therefore CB = \frac{6 \times R}{\frac{1}{2} R} = 12;$$

and the angle $B = 90^\circ - 60^\circ = 30^\circ$.

In the same manner the triangle might be solved, if *AB* and the angle *B* were given.

If *AC* and the angle *B* were given; since the angle $C = 90° - B$, *C* is also known, and ∴ this is reducible to the second case.

3. Having given the hypothenuse *BC*, and one side *AC*; to find the rest.

Here (72) $BC : CA :: \text{rad.} : \cos. C$, in which proportion the three first terms being known, $\cos. C = \frac{R \times CA}{BC}$ may be computed, and ∴ the angle *C* determined from the tables.

Also the angle $B = 90° - C$ and is ∴ known.

And $AB = \surd(BC^2 - CA^2) = \surd\{(BC + CA).(BC - CA)\}$ which may be found;

Or since $\text{rad.} : \sin. C :: BC : BA$, ∴ $BA = \frac{BC \times \sin. C}{R}$ may be computed.

In the same manner the triangle may be solved, if *BC* and *AB* be given.

4. Having given the two sides *BA* and *AC* containing the right angle; to find the rest.

Here (74) $AC : AB :: \text{rad.} : \text{tang.} C$, in which proportion the three first terms being known, the $\text{tang.} C = \frac{AB \times R}{AC}$ may be computed, and the angle *C* determined from the tables.

Also the angle $B = 90° - C$.

And tabular radius : $\sec. C :: AC : CB$, whence $CB = \frac{AC \times \sec. C}{R}$ may be determined;

or $CB = \dfrac{AC \times R}{\cos.\ C}$ may be computed.

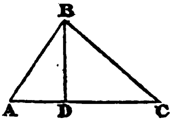
(77.) By Case 2. the height of any object may be determined, the base of which is accessible. From the base A, measure along the horizontal plane any length AC. At C let the angle be observed which the object subtends. Then its height $BA = \dfrac{AC \times \text{tang. } C}{R}$.

Prop. IX.

(78.) *The sides of any triangle are to one another as the sines of the angles opposite to them.*

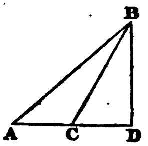
Let ABC be any triangle, BD a perpendicular let fall on AC, and

1. Let it fall within the triangle, then



(71) $AB : BD$ :: rad : sin. A,
and $BD : BC$ :: sin. C : rad.
∴ *ex æquo per*. $AB : BC$ :: sin. C : sin. A

2. But if BD falls without the triangle,

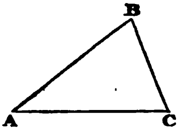


$AB : BD$ :: rad. : sin. A
and $BD : BC$ :: sin. BCD : rad.
∴ $AB : BC$ :: sin. BCD : sin. A
:: sin. BCA : sin. A, since the angle BCA is the supplement of BCD (20).

(79.) In any rectilineal triangle, it is sufficient if three out of the six parts which belong to it are known, provided that one of these parts be a side; for if only three angles be given, any triangle equiangular to the given one will satisfy the conditions.

Prop. X.

(80.) *Having given, in any triangle, a side and an angle opposite to it, and also one other angle or one other side, the remaining sides and angles may be found.*



1. Having given the angles at A and C, and the side BC; to find the rest.

Since the angles at A and C are given, their sines may be found from the tables, and (78) sin. A : sin. C :: BC : BA, in which proportion the three first terms being known, $BA = \dfrac{BC \times \sin. C}{\sin. A}$ may be determined.

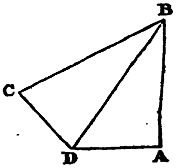
Also the angle $B = 180^\circ - (A+C)$ and may ∴ be found.

Whence also sin. A : sin. B :: BC : CA,

$$\text{and } \therefore \; CA = \frac{BC \times \sin. B}{\sin. A} \text{ may be computed.}$$

In the same manner, if the angles A and B and the side BC; or the angles B and C and the side AC; or the angles A and C and the side AC were given; the other sides and angles might be found.

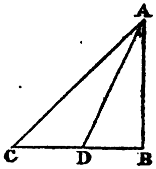
Ex. 1. Hence may be determined the height of an object, the base of which is inaccessible.



Let AB be the height; along the horizontal base measure any distance CD, and at C let the angle BCD be observed, and at D the angles BDC, BDA; then in the triangle BDC, the angle $CBD = 180^\circ - (BCD + BDC)$ and ∴ is known;

$$\text{and sin. } CBD : \text{sin. } BCD :: CD : DB,$$

$\therefore DB = CD \times \dfrac{\text{sin. } BCD}{\text{sin. } CBD}$, which may ∴ be computed. Hence in the right-angled triangle BAD, rad : sin. $BDA :: BD : BA$, and ∴ $BA = \dfrac{BD \times \text{sin. } BDA}{\text{rad.}}$ which may ∴ be determined.



Ex. 2. To determine the height of a cloud A or any object in the air; let two observers C and D in the same vertical plane take the angles of elevation ACB, ADB ($= a^0$ and b^0) and measure CD ($= c$ yards) the distance between them; the height and the distances from them may be found; for the angle ADB being $= b^0$, $ACB = a^0$, $ADC = 180^0 - b$, and $CAD = b^0 - a^0 = d$.

Hence in the triangle CAD, sin. CAD : sin. C :: $CD : DA$,

or sin. d : sin. a :: c : $DA = \dfrac{c\,.\,\text{sin. } a}{\text{sin. } d}$, which is ∴ known.

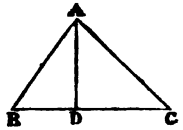
And sin. CAD : sin. CDA :: $CD : CA$,

i. e. sin. d : sin. $(180 - b)$ :: c : $CA = \dfrac{c\,.\,\text{sin } b}{\text{sin. } d}$, which is also known. Hence in the right-angled triangle ADB, rad. : sin. ADB :: $AD : AB$,

$$\therefore AB = \frac{AD \times \text{sin. } ADB}{R} = \frac{c\,.\,\text{sin. } a \times \text{sin. } b}{R\,.\,\text{sin. } d},$$

which may be determined.

Ex. 3. To determine the breadth of a river and the distance of an object B by its side from another object A



on the opposite bank, let a line BC be measured ($=a$ yards) along its bank, and by means of a Theodolite let the angles CBA, BCA be measured ($=b^\circ$ and c°); then the angle $BAC = 180^\circ - (B+C) = d^\circ$.

$$\text{And sin. } BAC : \text{sin. } C :: BC : AB,$$

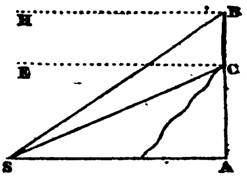
or $\text{sin. } d : \text{sin. } c :: a : AB = \dfrac{a \text{ sin. } c}{\text{sin. } d}$, which may be determined. Whence in the right-angled triangle ADB,

$$\text{rad.} : \text{sin. } B :: AB : AD,$$

$$\therefore AD = \frac{AB . \text{sin. } B}{R} = \frac{a . \text{sin. } c \times \text{sin. } b}{R . \text{sin. } d},$$

which may be computed.

Ex. 4. From B the top of a tower, the angle of depression of a vessel at anchor ($HBS = a^\circ$) was observed, and at C the bottom of the tower, the angle of depression ($ECS = b^\circ$) was again observed. The height of the tower ($BC = c$) being known; to determine the horizontal distance AS, and the height CA of the bottom of the tower above the level of the sea.



The angle $BSC = BSA - CSA = HBS - ECS = a - b = d$ and $SBC = 90 - HBS = 90 - a$,
$\therefore$ in the triangle SBC,
$\text{sin. } BSC : \text{sin. } SBC :: BC : CS$
or $\text{sin. } d : \text{cos. } a :: c : CS$,

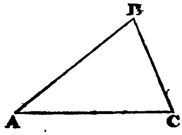
$$\therefore CS = \frac{c . \text{cos. } a}{\text{sin. } d}, \text{ which may be found.}$$

And in the triangle CSA, $\text{rad.} : \text{sin. } CSA :: CS : CA$,

$$\therefore CA = \frac{CS . \text{sin. } CSA}{R} = \frac{c . \text{cos. } a \times \text{sin. } b}{R \times \text{sin. } d},$$

and rad. : cos. CSA :: CS : SA,

$$\therefore SA = \frac{CS \times \cos. CSA}{R} = \frac{c . \cos. a \times \cos b}{R \times \sin. d}.$$



2. Having given the angle at A and the sides AC, CB; to find the rest.

The angle at A being given, its sine may be found from the tables; and (78) $CB : CA :: \sin. A : \sin. B$;

hence $\sin. B = \frac{CA}{CB} \times \sin. A$, and may $\therefore$ be determined. But as the sine of an angle is equal to the sine of its supplement, the angle B may be greater or less than a right angle, unless BC be greater than AC and consequently the angle A greater than the angle B*.

The angle B being found, the angle $C = 180 - (A + B)$, and may $\therefore$ be determined.

Also $\sin. A : \sin. C :: CB : BA$,

whence $BA = CB \times \frac{\sin. C}{\sin. A}$ may be computed.

In the same manner may be solved any case, wherein are given two sides and an angle opposite to one of them.

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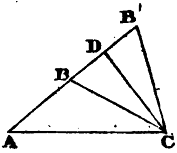
* This ambiguity occurs in oblique-angled triangles whenever the side opposite to the given angle is less than the side adjacent to it.



For if CB be less than AC, from C draw CD perpendicular to AB, produced if necessary. Make $DB' = DB$, and join CB'. (Eucl. i. 4.) $CB' = CB$ and the angle $CB'D$ is equal to CBD, which is the supplement of ABC. Hence there are two triangles CBA, $CB'A$, which have one angle and two sides answering the conditions.

(81.) If all the angles of any triangle be given, the ratio of the sides to each other may be found.

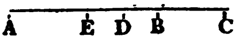
For the angles A, B and C being given, their sines may be found from the tables,

and (78) sin. C : sin. A :: AB : BC,
sin. B : sin. C :: AC : AB
sin. A : sin. B :: BC : AC,

in which proportions the two first terms being given, the ratio of the third to the fourth in each is given.

LEMMA III.

(82.) *If the semi-sum of two quantities be added to their semi-difference, the aggregate will be the greater of the two; if the semi-difference be subtracted from the semi-sum, the remainder will be the less of the two.*



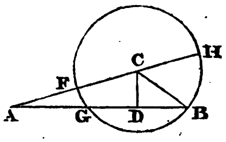
Let AB and BC representing the two quantities be placed in the same straight line, which bisect in D, and cut off $AE = BC$ and $\therefore DE = DB$. Then $AB - BC = AB - AE = 2DB$, whence $DB =$ the semi-difference of the lines; and since $AD =$ the semi-sum, $AD + DB = AB$ the greater, and $AD - DB = DC - DB = BC$ the less.

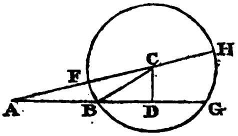
(83.) COR. If the semi-sum be subtracted from the greater quantity, the remainder will be the semi-difference.

For $AB - AD = DB$, the semi-difference.

PROP. XI.

(84.) *In any triangle, if a perpendicular from the vertex be drawn upon the base; the base will be to the* *sum of the sides as the difference of the sides to the difference or sum of the segments of the base made by the perpendicular, according as the perpendicular falls within or without the triangle.*





Let ABC be the proposed triangle, C the vertex, and AB the base; draw the perpendicular CD cutting the base or base produced in D. With the centre C, and radius CB the less of the two sides, describe a circle cutting the side AC in F, and the base or base produced in G; produce AC to H. Then AH is equal to the sum of the sides AC, CB; and AF to their difference. And because $BD = DG$, (Eucl. III. 3.) AG is the difference of the segments AD, DB in one case, and their sum in the other. And (Eucl. iii. 36.) the rectangle AB, AG is equal to the rectangle AH, AF,

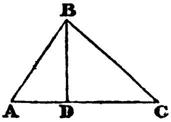
$$\therefore AB : AH :: AF : AG.$$

(85.) Cor. Hence if the three sides of any triangle are given, the three angles may be found. For the three sides being given, the three first terms in the preceding proportion are given, and consequently the fourth, *i. e.* the difference of the segments of the base made by the perpendicular when it falls within the triangle, may be found; and the base or sum of the segments is also given, $\therefore$ the segments themselves AD, DB may be found (82). Hence in each of the right-angled triangles ACD, BCD, the two sides being known, the angles at A and B may be determined (76), and $\therefore$ the remaining angle at C (13).

But if the perpendicular fall without the triangle, the fourth term in the proportion or sum of the segments, will be found; and the base AB, which is their difference, is given, ∴ the segments may be determined, and the angles as before.

Prop. XII.

(86.) *In any plane triangle, the cosine of an angle is to radius as the difference between the sum of the squares of the sides which contain that angle and the square of the third side, is to twice the rectangle contained by the two first sides.*



Let ABC be any triangle; from B let fall the perpendicular BD on AC; then (Eucl. ii. 13.)

$$BC^2 = AB^2 + AC^2 - 2AC \times AD,$$

$$\text{or } 2AC \times AD = AB^2 + AC^2 - BC^2.$$

$$\text{Now cos. } A : \text{rad.} :: DA : AB :: 2AC \times AD : 2AC \times AB$$
$$:: AB^2 + AC^2 - BC^2 : 2AC \times AB.$$

Nearly in the same manner it may be shewn, if the perpendicular falls without the triangle.

(87.) Cor. 1. Let a, b, c be the three sides opposite to the angles A, B, C; then cos. $A = \frac{b^2 + c^2 - a^2}{2bc}$,

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}, \text{ and cos. } C = \frac{a^2 + b^2 - c^2}{2ab}.$$

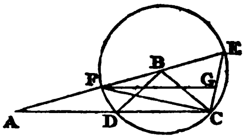
Hence the sides of a triangle being given, the angles may be found*.

---

* The preceding expressions not being easy for calculation, values may

Prop. XIII.

(88.) *In any triangle, the sum of any two sides is to their difference as the tangent of the semi-sum of the angles at the base is to the tangent of their semi-difference.*



Let *ABC* be any triangle, whose base is *AC*. With the centre *B*, and radius *BC* the less of the two sides, describe a circle meeting the side *AB* in *F* and *AB* produced in *E* and *AC* in *D*. Join *BD*, *CE*, *CF*; and draw *FG* parallel to *AC*.

Then since *CBE*, *CFE* are on the same base *CE*, (Eucl. iii. 20.) $CFE = \frac{1}{2} CBE = \frac{1}{2}(CAB + BCA)$; and since $BD = BC$, the angle $BCD = BDC$, and ∴ *DBA* is equal to the difference of the angles *BCA*, *BAC*. Hence (Eucl. iii. 20.) *FCD* and ∴ its equal $CFG = \frac{1}{2}(BCA - BAC)$. If now with the centre *F*, and radius *FC*, a circle be described,

$$CE : CG :: \text{tang. } CFE : \text{tang. } CFG.$$

But *FG* being parallel to *AC* one of the sides of the triangle *AEC*, (Eucl. vi. 2.)

$$CE : CG :: AE : AF :: AB + BC : AB - BC,$$

$$\therefore AB + BC : AB - BC :: \text{tang. } CFE : \text{tang. } CFG$$
$$:: \text{tang. } \tfrac{1}{2}(BCA + BAC) : \text{tang. } \tfrac{1}{2}(BCA - BAC).$$

---

may be deduced for the sines, $\sin^2 . A = 1 - \cos^2 . A = 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2$

$$= \frac{(a+b+c).(b+c-a).(a+b-c).(a+c-b)}{4b^2c^2},$$

and ∴ $\sin . A = \frac{1}{2bc} \times \sqrt{\{(a+b+c).(a+b-c).(a+c-b).(b+c-a)\}}$

$$= \frac{2}{bc} \times \sqrt{\{P.(P-a).(P-b).(P-c)\}}, \text{ if } 2P = a+b+c.$$

(89.) COR. 1. Hence if the sides *AB*, *BC* of any triangle, and the angle *ABC* included between those sides, be given, the other sides and angles may be found.

For, from the included angle *ABC*, its supplement *EBC* may be determined; half of which is the semi-sum of the angles at the base; and their semi-difference may be found by the Proposition. Having ∴ the semi-sum and semi-difference of the angles at the base, the angles themselves *BAC*, *BCA* may be found (82). The three angles and two sides, ∴ being obtained, the third side may be found, (78).

Or the side may be calculated by means of a subsidiary angle, thus

$$
\begin{aligned}
(87),\ b^2 &= a^2+c^2-2ac\,.\cos.\,B \\
&= (a-c)^2+2ac\,.\,(1-\cos.\,B) \\
&= (a-c)^2+4ac\,.\sin^2.\,\tfrac{1}{2}B\ (70) \\
&= (a-c)^2\times\left\{1+\frac{4ac\,.\sin^2.\,\frac{1}{2}B}{(a-c)^2}\right\}.
\end{aligned}
$$

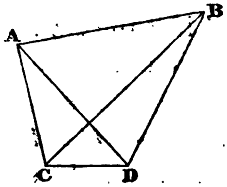
Now since tangents admit of every degree of magnitude, find an angle x whose tangent is $\frac{2\sqrt{ac}\,.\sin\frac{1}{2}B}{a-c}$, and

$\therefore b=(a-c)\,.\sqrt{(1+\text{tang}^2.\,x)}=(a-c)\,.\sec.\,x=\frac{a-c}{\cos.\,x}$,

or to radius $R=\frac{(a-c)\,.R}{\cos.\,x}$ which (since x is known from the equation $\text{tang}.\,x=\frac{2\sqrt{ac}\,.\sin.\,\frac{1}{2}B}{a-c}$), may easily be computed.

Ex. To determine the distance between two visible inaccessible objects.

Let *A* and *B* be the two objects. Measure a given line *CD* ($=a$) along a horizontal base, from the extre-



mities of which A and B are visible. At C observe the angles ACB, BCD, and at D the angles ADC, ADB. Then in the triangle ACD, the angle $CAD = 180° - (ACD + ADC) = b$, and $\sin. DAC : \sin. ADC :: CD : CA$

$$\text{or } \sin. b : \sin. c :: a : CA,$$

$$\therefore\ CA = \frac{a\,.\,\sin.\,c}{\sin.\,b}, \text{ which may be computed.}$$

And in the triangle BCD, the angle $CBD = 180° - (CDB + BCD) = d$, whence

$$\sin. CBD : \sin. CDB :: CD : CB;$$

$$\text{or } \sin. d : \sin. e :: a : CB,$$

$$\therefore\ CB = \frac{a\,.\,\sin.\,e}{\sin.\,d}, \text{ which may be computed.}$$

And in the triangle ACB, $AC + CB : AC \sim CB$
$:: \text{tang. } \tfrac{1}{2}(BAC + ABC) : \text{tang. } \tfrac{1}{2}(BAC \sim ABC)$
$:: \text{tang. } \tfrac{1}{2}(180 - ACB) : \text{tang. } \tfrac{1}{2}(BAC \sim ABC)$,

$$\therefore\ \text{tang. } \tfrac{1}{2}(BAC \sim ABC) = \frac{AC \sim CB}{AC + CB} \times \text{cot. } \tfrac{1}{2} ACB,$$

which may be calculated. Hence from the tables the difference of the angles may be found, and the sum being known, the angles themselves may be determined (82). And $\sin. ABC : \sin. ACB :: AC : AB$

$$\therefore\ AD = \frac{\sin. ACB}{\sin. ABC} \times AC \text{ may be calculated.}$$

Or AB may be ascertained as before by means of the subsidiary angle x; and from the three sides being known the two remaining angles may be determined.

(90.) COR. 2. Hence also the sum of the sines of two angles is to the difference of the sines, as the tangent of half the sum of those angles is to the tangent of half their difference.

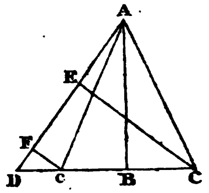
For $AC : BC :: \sin. B : \sin. A$,

$\therefore AC+BC : AC\sim BC :: \sin. B+\sin. A : \sin. B\sim\sin. A$

and $\therefore \sin. A+\sin. B : \sin. A\sim\sin. B :: \text{tang.}\frac{1}{2}(A+B)$
[$: \text{tang.}\frac{1}{2}(A\sim B)$.

Prop. XIV.

(91.) *The sum of the tangents of any two angles is to their difference as the sine of their sum is to the sine of their difference.*



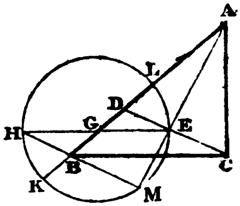
Let *DAB*, *BAC* be the angles, and through any point *B* in *AB*, draw *DC* perpendicular to *AB*. Make the angle $BAc = BAC$; then will $Ac = AC$. From *C* and *c* draw *CE*, *cF* perpendicular to *AD*.

Then $CE : cF :: \sin. CAD : \sin. cAD$
$:: \sin. (CAB+BAD) : \sin. (BAD\sim BAC)$.
And $BD : BC :: \text{tang.} BAD : \text{tang.} BAC$,
$\therefore CD : cD :: \text{tang.} BAD+\text{tang.} BAC :$
[$\text{tang.} BAD\sim\text{tang} BAC$.

And *cF*, *CE* being perpendicular to *AD*, are parallel,
whence $CE : cF :: CD : cD$
or $\sin. (DAB+BAC) : \sin. (DAB\sim BAC) ::$
$\text{tang.} DAB+\text{tang.} BAC : \text{tang.} DAB\sim\text{tang.} BAC$.

Prop. XV.

(92.) *In any plane triangle, the rectangle contained by any two sides is to the rectangle contained by the* *excesses of half the perimeter above each of them respectively as the square of radius is to the square of the sine of half the angle contained by those sides.*



Let ABC be any triangle, BC its base, AB the greater of the sides and AC the less; $P = \frac{1}{2}$ the perimeter of the triangle, then $AB \times AC : (P - AB) . (P - AC) :: R^2 : \sin^2 . \frac{1}{2} A$.

In AB take $AD = AC$, join DC; and draw AE perpendicular to DC, and EG parallel to BC, cutting AB in G. With the centre G, and radius GE, describe a circle cutting AB in L, and AB and EG produced in K and H. Join HB; and produce AE, HB till they meet in M.

Since $AD = AC$, and the angles at E are right angles, the squares of AE, ED are equal to the squares of AE, EC; and $\therefore ED = EC$, or $DC = 2DE$. Hence by similar triangles DGE, DBC, $BC = 2GE = EH$, and BC is also parallel to EH, $\therefore HBM$ is parallel to CED, and (Eucl. i. 29.) the angle $BME = DEA$, or it is a right angle; and HE being a diameter, M is a point in the circle.

Now (71) $AB : BM ::$ rad. : (sin. $DAE =$) sin. $\frac{1}{2} BAC$,

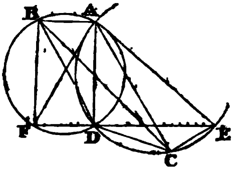
and $AD : DE ::$ rad. : sin. $\frac{1}{2} BAC$,

$\therefore$ *comp.* $AB \times AD : BM \times DE :: R^2 : \sin^2 . \frac{1}{2} BAC$.

But from the similar triangles DGE, DBC, $DB = 2DG$, to each of these equals $AD + AC = 2AD$, and $BA + AC = 2AG$, to each of which equals add $BC = 2GE = 2GK$, $\therefore AB + AC + BC = 2AK$, $\therefore AK = P$, and $BK = AK - AB = P - AB$, and $BL = DK = AK - AD = P - AC$. And the rectangle $BM \times DE = BM \times EC = BM \times BH$

= (Eucl. iii. 35.) $BK \times BL = (P - AB).(P - AC)$, hence $AB \times AC : (P - AB).(P - AC) :: R^2 : \sin^2.\frac{1}{2} BAC$.

(93.) Cor. 1. Hence if the three sides of any triangle be given, the angles may be found. For if the sides opposite to the angles A, B, C be a, b, c, respectively, then $\sin^2.\frac{1}{2} A = r^2 \times \frac{(p-c).(p-b)}{cb}$, may be determined, or the sine of half the vertical angle, and consequently the vertical angle itself may be found. Hence the other angles (78).



Ex. Three points A, B, C being given in a plane, to determine the position of a fourth D, where the angles ADB, ADC, subtended by A and B, A and C are $a°$ and $b°$ respectively.

On AB describe a segment of a circle containing an angle $a°$, and on AC a segment containing an angle $b°$; the point of intersection D is the point required. Let $AB = c$, $AC = d$, $BC = e$, $\sin.\frac{1}{2} BAC = \frac{R\sqrt{(p-c).(p-d)}}{\sqrt{cd}}$, which may be computed, p being $= \frac{1}{2}.(c+d+e)$. Draw the diameter AF, and join FB. In the triangle BAF, $AF = \frac{BA \times R}{\sin.\, BFA} = \frac{c \times R}{\sin.\, a}$ may be found. Draw the diameter AE, and join CE. In the triangle ACE, the angle $CAE = ADC - ADE = b - 90° = g$, and $AE = \frac{R \times AC}{\cos.\, CAE} = \frac{R.AC}{\cos.\, g}$. Join DE, DF, they are in the same straight line (ii. 5). Hence in the triangle FAE, AE and AF having been computed, and the angle $FAE = BAC + CAE - FAB = h$, and tang. $\frac{1}{2}(EFA \sim AEF) =$

$\frac{AE \sim AF}{AE+AF} \times$ cotang. FAE; whence the angle AFE may be determined. And in the right-angled triangle DAF, $AD = \frac{AF \times \text{sin. } AFD}{\text{rad.}}$ may be found; which line and the angle BAD determine the position of D.

(94.) Cor. 2. Also $AB : AM :: \text{rad.} : \text{cos. } \tfrac{1}{2} BAC$,
and $AD : AE :: \text{rad.} : \text{cos. } \tfrac{1}{2} BAC$,
$\therefore AB \times AD : AM \times AE :: R^2 : \text{cos}^2. \tfrac{1}{2} BAC$.
and since $AD = AC$, and $AM \times AE = AK \times AL = P \times$
[$(P-BC)$

$$AB \times AC : P.(P-BC) :: R^2 : \text{cos}^2. \tfrac{1}{2} BAC,$$

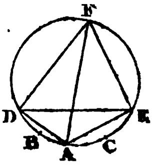
$$\therefore \text{cos. } \tfrac{1}{2} A = r . \sqrt{\left\{\frac{p \times (p-a)}{bc}\right\}}.$$

(95.) Cor. 3. Also $AE : ED :: R : \text{tang. } \tfrac{1}{2} BAC$,
and $AM : MB :: R : \text{tang. } \tfrac{1}{2} BAC$,
$\therefore AE \times AM : ED \times MB :: R^2 : \text{tang}^2. \tfrac{1}{2} BAC$,
or $P.(P-BC) : (P-AB).(P-AC) :: R^2 : \text{tang}^2.$
[$\tfrac{1}{2} BAC$,

$$\therefore \text{tang. } \tfrac{1}{2} A = r . \sqrt{\left\{\frac{(p-b).(p-c)}{p.(p-a)}\right\}}.$$

Prop. XVI.

(96.) *Having given the sines and cosines of two arcs; to find the sine of an arc which is equal to their sum.*



Let AB, AC be the two arcs; take AD, AE their doubles; draw the diameter AF. Join AD, DF, AE, EF, ED.

Then the chord $AD = 2 . \text{sin. } AB$ (62)
$AE = 2 . \text{sin. } AC$,
$DE = 2 . \text{sin. } BC$,

and $DF = 2 \cos. \tfrac{1}{2} AD = 2 \cos. AB$ (70),
$EF = 2 \cos. AC.$

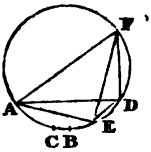
Now $AF \times ED = AD \times FE + AE \times FD$ (Eucl. vi. D.)
or $2 \text{ rad.} \times 2 \sin. BC = 2 \sin. AB \times 2 \cos. AC$
$+ 2 \sin. AC \times 2 \cos. AB,$
$\therefore \text{rad} \times \sin. BC = \sin. AB \times \cos. AC + \sin. AC \times \cos. AB.$

(97.) Cor. 1. If rad. $= 1$, $AB = a$, $AC = b$,
$\sin. (a + b) = \sin. a \times \cos. b + \sin. b \times \cos. a.$

(98.) Cor. 2. If $a = b$, $\sin 2a = 2 \sin. a \times \cos. a.$
Hence also $\sin. 3a = 3 \sin. a - 4 \sin^3. a$
$\sin. 4a = (4 \sin. a - 8 \sin^3. a) . \cos. a$
$\sin. 5a = 5 \sin. a - 20 \sin.^3 a + 16 \sin^5. a.$

Prop. XVII.

(99.) *Given the sines and cosines of two arcs; to find the sine of an arc which is equal to their difference.*

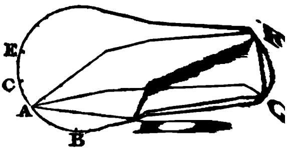


Let AB, AC be the two arcs; take AD, AE the doubles of them; draw the diameter AF;
join AE, AD, FE, FD, DE.
Then the chord $AD = 2 . \sin. AB$ (62)
$AE = 2 . \sin. AC$
$DE = 2 . \sin. BC,$
and $FD = 2 . \cos AB$ (70)
$EF = 2 . \cos. AC;$
and $AF \times ED = AD \times EF - AE \times FD$ (Eucl. vi. D.)
or $2 \text{ rad.} \times 2 \sin. BC = 2 \sin. AB \times 2 \cos. AC -$
[$2 \sin. AC \times 2 \cos. AB,$
$\therefore \text{rad.} \times \sin. BC = \sin. AB \times \cos. AC - \sin. AC \times \cos. AB.$

(100.) Cor. If rad. $= 1$, and $AB = a$, $AC = b$,
$\sin. (a - b) = \sin. a \times \cos. b - \sin. b \times \cos. a.$

Prop. XVIII.

(101.) *Given the sines and cosines of two arcs to find the cosine of an arc which is equal to their sum.*



Let AB, AC be the two arcs. Take AD, AE the doubles of them. Draw the diameter AF, and make $FG = AE$. Join FG, GD, DA, AG, FD. Then GD is the supplement of DE, and $\therefore$ its

chord $DG = 2 \text{ cos. } \frac{1}{2} DE = 2 \text{ cos. } BC$ (70).

$DF = 2 \text{ cos. } AB$,

and $AG = 2 \text{ cos. } \frac{1}{2} FG = 2 \text{ cos } \frac{1}{2} AE = 2 \text{ cos. } AC$.

$AD = 2 \text{ sin. } AB$,

and $FG = 2 \text{ sin. } \frac{1}{2} FG = 2 \text{ sin. } AC$.

And $AF \times DG = FD \times AG - AD \times FG$ (Eucl. vi. D.)

or $2 \text{ rad.} \times 2 \text{ cos. } BC = 2 \text{ cos. } AB \times 2 \text{ cos. } AC - 2 \text{ sin. } AB \times 2 \text{ sin. } AC$,

$\therefore \text{rad.} \times \text{cos. } BC = \text{cos. } AB \times \text{cos. } AC - \text{sin. } AB \times \text{sin. } AC$.

(102.) Cor. 1. If rad. $= 1$, $AB = a$, $AC = b$,

$\text{cos. } (a+b) = \text{cos. } a \times \text{cos. } b - \text{sin. } a \times \text{sin. } b$.

(103.) Cor. 2. If $a = b$, $\text{cos. } 2a = \text{cos}^2 a - \text{sin}^2. a$

$= 2 \text{ cos}^2. a - 1$,

or $\text{cos. } 2a = 1 - 2 \text{ sin}^2. a$.

Also $\text{cos. } 3a = 4 \text{ cos}^3. a - 3 \text{ cos. } a$

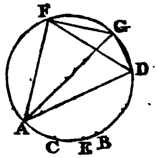
$\text{cos. } 4a = 8 \text{ cos}^4. a - 8 \text{ cos}^2. a + 1$.

$\text{cos. } 5a = 16 \text{ cos}^5. a - 20 \text{ cos}^3. a + 5 \text{ cos. } a$

$\text{cos. } 6a = 32 \text{ cos}^6. a - 48 \text{ cos}^4. a + 18 \text{ cos}^2. a - 1$.

Prop. XIX.

(104.) *Given the sines and cosines of two arcs; to find the cosine of an arc which is equal to their difference.*



Let AB, AC be the two arcs, take AD, AE the doubles of them; draw the diameter AG, and make GF (on the opposite side) $= AE$. Join AD, AF, DF, DG, FG. The arc DGF is the supplement of DE, which ∴ is equal to $AD - AE = 2AB - 2AC = 2BC$, and ∴ $DF = 2$ cos. $\frac{1}{2}DE = 2$ cos. BC.

$DG = 2$ cos. AB (70) $AD = 2 .$ sin. AB (62.)

$AF = 2$ cos. AC $FG = 2 .$ sin. AC.

Now $AG \times DF = DG \times AF + AD \times GF$ (Eucl. vi. D.)
or 2 rad. × 2 cos. $BC = 2$ cos. $AB \times 2$ cos. $AC + 2$ sin. $AB \times 2$ sin. AC.

∴ rad. × cos. $BC =$ cos. $AB \times$ cos. $AC +$ sin. $AB \times$ sin. AC,

(105.) Cor. If rad. $= 1$, $AB = a$, $AC = b$,
cos. $(a - b) =$ cos. $a \times$ cos. $b +$ sin. $a \times$ sin. b.

(106.) It may perhaps be objected that the preceding propositions are proved only when the arcs (a) and (b) or even $(a + b)$ are less than quadrants. Assuming them to be proved within such a limit that (a) does not exceed a value α, and (b) a value β, it may be proved by means of what has been shewn before, that the values of the sines and cosines of the sums are equally true, when (a) does not exceed $90^\circ + \alpha$ and (b) does not exceed β.

For let $a = 90^\circ + m$, ∴ $m = a - 90^\circ$, which is less than 90°.

Now sin. $(90^\circ + m + b) =$ sin. $\{90^\circ - (m + b)\}$
$=$ cos. $(m + b)$
$=$ cos. $m \times$ cos. $b -$ sin. $m \times$ sin. b
$=$ sin. $(90^\circ - m) \times$ cos. $b -$ cos. $(90^\circ - m) \times$ sin. b
$=$ sin. $(180^\circ - a) \times$ cos. $b -$ cos. $(180^\circ - a) \times$ sin. b
$=$ sin. $a \times$ cos. $b +$ cos. $a \times$ sin. b.

Also $\cos.\ (90+m+b) = -\cos.\ \{90-(m+b)\}$

$= -\sin.\ (m+b)$

$= -\sin.\ m \times \cos.\ b - \cos.\ m \times \sin.\ b$

$= -\cos.\ (90-m) \times \cos.\ b - \sin.\ (90-m) \times \sin.\ b$

$= -\cos.\ (180-a) \times \cos.\ b - \sin.\ (180-a) \times \sin.\ b$

$= \cos.\ a \times \cos.\ b - \sin.\ a \times \sin.\ b.$

Hence ∴ these expressions which were demonstrated for (a) less than α and (b) less than β, are also true when (a) does not exceed $90+\alpha$ and (b) does not exceed β. In the very same manner from the preceding, it might be proved that they are true, when (a) does not exceed $90+\alpha$ and (b) $90+\beta$, and so on, *i.e.* they are true whatever be the values of (a) and (b) The same is equally applicable to the values of $\sin.\ (a-b)$ and $\cos.\ (a-b)$.

(107.) Since $\sin.\ (30^0+a) = \sin.\ 30^0 \times \cos.\ a + \cos.\ 30^0 \times \sin.\ a$,

and $\sin.\ (30^0-a) = \sin.\ 30^0 \times \cos.\ a - \cos.\ 30^0 \times \sin.\ a$,

∴ $\sin.\ (30^0+a) + \sin.\ (30^0-a) = 2\sin.\ 30^0 \times \cos.\ a = \cos.\ a$ [(63).

and $\sin.\ (30^0+a) = \cos.\ a - \sin.\ (30^0-a)$. If ∴ the sines and cosines of all arcs less than 30° be known, the sines of all arcs above 30° and less than 60° might be found by subtraction.

(108.) Since $\sin.\ (60^0+a) = \sin.\ 60^0 \times \cos.\ a + \cos.\ 60^0 \times \sin.\ a$,

and $\sin.\ (60^0-a) = \sin.\ 60^0 \times \cos.\ a - \cos.\ 60^0 \times \sin.\ a$,

∴ $\sin.\ (60^0+a) - \sin.\ (60^0-a) = 2\cos.\ 60^0 \times \sin.\ a = \sin.\ a$,

and $\sin.\ (60^0+a) = \sin.\ a + \sin.\ (60^0-a)$. If ∴ the sines of arcs less than 60° be known, the sines of arcs greater than 60° may be found, by addition.

(109.) Let $a = nA$ and $b = A$.

Then $\sin. \{(n+1).A\} = \sin. nA \times \cos. A + \cos. nA \times \sin. A$,

and $\sin. (n-1).A = \sin. nA \times \cos. A - \cos. nA \times \sin. A$,

$\therefore \sin. (n+1).A + \sin. (n-1).A = 2 \sin. nA \times \cos. A$[*],

and $\sin. (n+1).A = 2 \sin. nA \times \cos. A - \sin. (n-1).A$.

Also $\sin. (n+1).A = \sin. (n-1).A + 2 \sin. A \times \cos. nA$,

and $\cos. (n+1).A = 2 \cos. nA \times \cos. A - \cos. (n-1).A$

or $= \cos. (n-1).A - 2 \sin. nA \times \cos. A$.

(110.) If in the equations (97) and (100)

$a + b = A$ and $a - b = B$, then $a = \frac{1}{2}(A+B)$ and $b = \frac{1}{2}(A-B)$

and $\sin. (a+b) = \sin. a \times \cos. b + \cos. a \times \sin. b$

$\sin. (a-b) = \sin. a \times \cos. b - \cos. a \times \sin. b$,

$\therefore \sin. (a+b) + \sin. (a-b) = 2 \sin. a \times \cos. b$

and $\sin. (a+b) - \sin. (a-b) = 2 \cos. a \times \sin. b$,

$\therefore \sin. A + \sin. B = 2 . \sin. \frac{1}{2}(A+B) \times \cos. \frac{1}{2}(A-B)$

and $\sin. A - \sin. B = 2 . \cos. \frac{1}{2}(A+B) \times \sin. \frac{1}{2}(A-B)$

Hence also $\sin. A + \sin. B = \dfrac{\cos. \frac{1}{2}(A-B)}{\cos. \frac{1}{2}(A+B)} \times \sin. (A+B)$,

and $\sin. A - \sin. B = \dfrac{\sin. \frac{1}{2}(A-B)}{\sin. \frac{1}{2}(A+B)} \times \sin. (A+B)$.

Hence also $\sin^2. A - \sin^2. B = \sin. (A+B) \times \sin. (A-B)$.

---

[*] Hence if a series of arcs be taken in arithmetic progression, radius is to twice the cosine of the common difference as the sine of any one arc taken as a mean is to the sum of the sines of any two equidistant extremes.

Also radius is to twice the cosine of the common difference as the cosine of any one arc taken as a mean is to the sum of the cosines of any two equidistant extremes.

And in a similar manner,

$$\cos^2. B - \cos^2. A = \sin. (A+B) \times \sin. (A-B).$$

(111.) To find the sin. $(A+B+C)$.

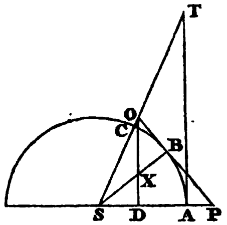
$$\text{Sin.} (A+B+C) = \sin. (A+B) \times \cos. C + \cos. (A+B) \times \sin. C$$

$$= \sin. A \times \cos. B \times \cos. C + \sin. B \times \cos. A \times \cos. C$$

$$+ \sin. C \times \cos. A \times \cos. B - \sin. A \times \sin. B \times \sin. C.$$

Prop. XX.

(112.) *Having given the tangents of two arcs; to determine the tangents of two arcs which may be equal to their sum and difference.*



Let AB, BC be the given arcs, AB being the greater, then AC is the sum of the arcs (in Fig. 1.) and their difference (in Fig. 2.); AT the tangent of their sum (1) or difference (2), BP, BO the tangents of the respective arcs AB, BC.

Draw OD perpendicular to SA cutting SB or SB produced in x, $\therefore$ the triangle OBx is similar to SxD and $\therefore$ to SBP; hence $OB : Bx :: SB : BP$, $\therefore OB \times BP = SB \times Bx$. Also by the similar triangles TAS, ODS,

$$AT : OD :: SA : SD$$

$$OD : OP :: SD : Sx,$$

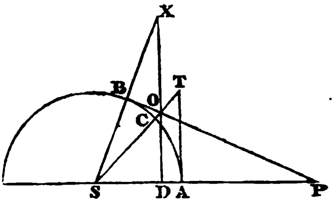
by the sim. triangles SxD, ODP,

$$\therefore AT : OP :: SA : Sx$$

$$:: SA : SB - Bx \text{ (fig. 1)}$$

$$:: SA^2 : SB^2 - SB \times Bx$$

$$:: SA^2 : SA^2 - OB \times BP.$$



Again (in Fig. 2.) $AT : OP$

$:: SA : Sx$

$:: SA : SB+Bx$

$:: SA^2 : SB^2+SB\times Bx$

$:: SA^2 : SA^2+OB\times BP.$

$$\text{Hence } AT = SA^2 \times \frac{OP}{SA^2 \mp OB\times BP}$$

$$= SA^2 \times \frac{OB \pm BP}{SA^2 \mp OB\times BP}.$$

(113.) Cor. 1. If rad. $= 1$, $AB = a$, $BC = b$,

$$\text{tang.}\,(a\pm b) = \frac{\text{tang.}\,a \pm \text{tang.}\,b}{1\mp\text{tang.}\,a\times\text{tang.}\,b}\ *.$$

(114.) Cor. 2. If $a=b$, $\text{tang.}\,2a = \dfrac{2\,.\,\text{tang.}\,a}{1-\text{tang}^2.\,a}$.

$$\text{Tang.}\,3a = \frac{3\,.\,\text{tang.}\,a-\text{tang}^3.\,a}{1-3\,.\,\text{tang}^2.\,a},$$

$$\text{tang.}\,4a = \frac{4\,.\,\text{tang.}\,a-4\,\text{tang}^3.\,a}{1-6\,\text{tang}^2.\,a+\text{tang}^4.\,a},$$

$$\text{tang.}\,5a = \frac{5\,\text{tang.}\,a-10\,\text{tang}^3.\,a+\text{tang}^5.\,a}{1-10\,\text{tang}^2.\,a+5\,\text{tang}^4.\,a},$$

$$\text{tang.}\,6a = \frac{6\,\text{tang.}\,a-20\,\text{tang}^3.\,a+6\,\text{tang}^5.\,a}{1-15\,\text{tang}^2.\,a+15\,\text{tang}^4.\,a-\text{tang}^6.\,a},\ \&c.$$

(115.) If (in Art. 113.) $a = 45°$,

$$\text{tang.}\,(45°+b) = \frac{1+\text{tang.}\,b}{1-\text{tang.}\,b},$$

$$*\ \text{Tang.}\,(a+b) = \frac{\sin.\,(a+b)}{\cos.\,(a+b)} = \frac{\sin.\,a\times\cos.\,b+\cos.\,a\times\sin.\,b}{\cos.\,a\times\cos.\,b-\sin.\,a\times\sin.\,b}$$

$$= \frac{\dfrac{\sin.\,a}{\cos.\,a}+\dfrac{\sin.\,b}{\cos.\,b}}{1-\dfrac{\sin.\,a\times\sin.\,b}{\cos.\,a\times\cos.\,b}} = \frac{\text{tang.}\,a+\text{tang.}\,b}{1-\text{tang.}\,a\times\text{tang.}\,b}.$$

$$\text{and tang. } (45 - b) = \frac{1 - \text{tang. } b}{1 + \text{tang. } b};$$

$$\therefore \text{tang. } (45^\circ + b) - \text{tang. } (45^\circ - b) = \frac{1 + \text{tang. } b}{1 - \text{tang. } b} - \frac{1 - \text{tang. } b}{1 + \text{tang. } b}$$

$$= 2 \,.\, \frac{2 \text{ tang. } b}{1 - \text{tang}^2 .\ b} = 2 \text{ tang. } 2b,$$

$$\text{and tang. } (45 + b) = \text{tang. } (45 - b) + 2 \text{ tang. } 2b.$$

If ∴ the tangents of arcs less than 45° be known, the tangents of arcs greater than 45° may be determined by addition.

(116.) Tang. a + cotang. $a = 2$ cosec. $2a$.

For tang. a + cotang. $a = \dfrac{\text{sin. } a}{\text{cos. } a} + \dfrac{\text{cos. } a}{\text{sin. } a} =$

$$\frac{\text{sin}^2 .\ a + \text{cos}^2 .\ a}{\text{sin. } a \times \text{cos. } a} = \frac{2}{\text{sin. } 2a} = 2 \text{ cosec. } 2a.$$

In the same way it may be shewn that

$$\text{cotang. } a - \text{tang. } a = 2 \text{ cotang. } 2a.$$

$$\text{and sec. } a + \text{tang. } a = \text{tang. } (45^\circ + \tfrac{1}{2} a)$$

$$\text{that tang. } a = \frac{\text{sin. } 2a}{1 + \text{cos. } 2a}, \text{ cotang. } a = \frac{\text{sin. } 2a}{1 - \text{cos. } 2a},$$

$$\text{cos. } 2a = \frac{1 - \text{tang}^2 .\ a}{1 + \text{tang}^2 .\ a}, \text{ and sec. } 2a = \frac{\text{sec}^2 .\ a}{1 - \text{tang}^2 .\ a}.$$

PROP. XXI.

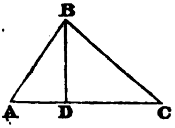
(117.) *The base and altitude of a triangle being given; to find its area.*

(Eucl. I. 41.) The area of the triangle $ABC = \frac{1}{2} AC \times BD$. (See next Fig.).

(118.) COR. 1. If a = the altitude BD, and b = the base *AC*, the area $S=\frac{ab}{2}$ and $a=\frac{2S}{b}$ and $b=\frac{2S}{a}$; any two of the terms ∴ being known, the third may be found.

PROP. XXII.

(119.) *Two sides of a triangle, and the included angle being given; to find its area.*



Let *AB*, *AC* and the angle *BAC* be given. Let fall the perpendicular *BD*,

$$AB : BD :: \text{rad.} : \sin. A,$$

$$\therefore BD = \frac{AB \times \sin. A}{R} = AB \times \sin. A,$$

(if $R = 1$),

$$\therefore \text{the area} = \tfrac{1}{2} AC \times BD = \tfrac{1}{2} AB \times AC \times \sin. A.$$

(120.) COR 1. Hence the areas of triangles which have one angle in each equal, are as the products of the sides containing those angles. Which is also true of parallelograms.

(121.) COR. 2. $AC=\frac{2S}{AB \times \sin. A}$, $AB=\frac{2S}{AC \times \sin. A}$,

$$\text{and } \sin. A = \frac{2S}{AB \times AC}.$$

PROP. XXIII.

(122.) *Given two angles and a side of a triangle; to find its area.*

Two angles being given, the third is also known.

Let *AC* be the given side.

Then $\sin. B : \sin. C :: AC : AB = \frac{AC \times \sin. C}{\sin. B}$, and $BD = AB \times \sin. A = \dfrac{AC \times \sin. A \times \sin. C}{\sin. B}$;

whence the area $= \frac{1}{2} AC^2 \times \dfrac{\sin. A \times \sin. C}{\sin. B}$

(123.) Cor. If the angles ABC, ACB be equal, the area will be $= \frac{1}{2} AC^2 \times \sin. A$. If the angles A and C are equal, the area will be $= \frac{1}{2} AC^2 \times \dfrac{\sin^2. A}{\sin. B}$.

But since the angle $B = 180^\circ - 2A$,

$\sin. B = \sin. 2A = 2 \sin. A \times \cos. A$,

whence the area $= \frac{1}{4} AC^2 \times \dfrac{\sin^2. A}{\sin. A \times \cos. A}$

$$= \tfrac{1}{4} AC^2 \times \frac{\sin. A}{\cos. A} = \tfrac{1}{4} AC^2 \times \text{tang. } A.$$

Prop. XXIV.

(124.) *Given the three angles, and the altitude of a triangle; to find its area.*

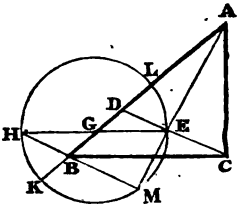
Since (122) $BD \times \sin. B = AC \times \sin. A \times \sin. C$, and $AC = \dfrac{2S}{BD}$, $\therefore 2S \times \sin. A \times \sin. C = BD^2 \times \sin. B$,

or $S = \frac{1}{2} BD^2 \times \dfrac{\sin. B}{\sin. A \times \sin. C}$.

Prop. XXV.

(125.) *Given the three sides of a triangle; to find its area.*

The area of the triangle ABC = area ADC + area BDC $= AE \times DE + ME \times DE$, ($BM$ being parallel to DC) $= AM \times DE$.



But by similar triangles ADE, AMB,

$$AE : ED :: AM : MB,$$

$$\therefore AE \times AM : ED \times AM :: ED \times AM : ED \times MB,$$

i. e. the area of the triangle is a mean proportional between $AE \times AM$ and $ED \times MB$.

$$\text{Now (92.) } ED \times MB = (P - AB).(P - AC),$$

$$\text{and } AE \times AM = AL \times AK = P \times (P - BC),$$

$$\therefore \text{ the } \overline{\text{area}}|^{2} = (P - AB).(P - AC).P.(P - BC)$$

$$\text{or } S = \sqrt{\{P.(P - AB).(P - AC).(P - BC)\}}.$$

(126.) If $AB = BC = AC$, $S = \frac{1}{4} AB^{2}.\sqrt{3}$, the area of an equilateral triangle.

If $BC = CA$, $S = \frac{1}{4} AB.\sqrt{\{(2BC + BA).(2BC - BA)\}}$, the area of an isosceles triangle.

(127.) To construct the trigonometrical canon.

It has been proved (103) that $2 \cos^{2}. A = 1 + \cos. 2A$; and therefore if the cosine of any arc be known, the cosine of half that arc may be determined. Now the sine of 30° has been found to be $= \frac{1}{2}$, $(R = 1)$, and the cosine $= \frac{\sqrt{3}}{2}$; if then in the formula $\cos. A = \sqrt{\{\frac{1}{2}.(1 + \cos. 2A)\}}$, $A = 15°$, cos. 15° may be determined. In the same manner from cos. 15°, the cos. 7° 30′ may be deduced; and so on, till after 11 divisions, cos. $52'' \, 44''' \, 3^{iv} \, 45^{v}$ is found; from which the sine of this arc may be determined. But from the nature of the circle, when the arc is very small, the ratio of the arc to the sine approaches nearly to a ratio of equality, $\therefore 52'' \, 44''' \, 3^{iv} \, 45^{v} : 1' ::$ the sine of the former arc : the sine of 1′; which $\therefore$ may be determined.

The sine and cosine of 1′ being ascertained, the sines of 2′, 3′, 4′, &c. may be determined (109) by making $n = 1, 2, 3,$ &c. and the cosines from (58). In this manner the sines and cosines of arcs as far as 30° may be computed. When the arc exceeds 30°, the sines may be computed by Art. (107), and the cosines as before, till the arc is 45°. And since the sine of an arc is equal to the cosine of its complement, the sines and cosines of arcs as far as 90° are determined. Also since the sines and cosines of arcs are equal to the sines and cosines of their supplements; the sines and cosines of all arcs up to 180° are known.

Since tang. $A = \frac{\text{sin. } A}{\text{cos. } A}$, the tangents of all arcs may be computed. When however they exceed 45°, they are more readily computed from (115) by addition. And the tangent of an arc being equal to the tangent of its supplement, the tangents of all arcs may be determined.

Hence also the cotangents (45).

Also the sec. $A = \frac{1}{\text{cos. } A}$, and ∴ the cosines being known, the secants may be determined. And the secants being known, the cosecants are also determined.

Also the versed sine ($= 1 \mp$ cosine) may be determined.

◆

ERRATA.

Page 17. line 3, *for* finite *read* indefinite.
19. — 3, *for* *ACB* *read* *ACE*.
37. — 20, *for* parts *read* points.
38. — 20, *for* centre *read* centres.
45. — 14, *for* circle *read* circles.
46. In the figure join *DE*.
47. — 18, *for* circumference *read* circumferences.
63. — 6 from the bottom, *for* 63 *read* 62.
64. — 13, *for* 63, *read* 62.
— line 4 from the bottom, *for* *OC* *read* *OB*.
76. In the figure join *DB*.
78. — 11, *for* made by the perpendiculars *read* cut off by the perpendiculars.
90. In the figure join *BC*.
108. — 7, *for* *AFC* *read* *AFG*.
109. — 6, *for* *A* *read* *B*.
112. — 12, *for* *Bl* *read* *BL*.
118. In the figure join *KG*.
122. last line, *for* a side produced *read* a side and a side produced.
123. — 7, *for* *AG* *read* *AC*.
149. line 3 from the bottom, *for* rectangle *read* rectangles.
164. — 16, *for* 2 *read* 62.
217. Join *CG* in the figure.
245. Fig. 2, join *DB*.
252. — 7, *for C read O*.
— line 10, *for AD read CD*.

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AN

HISTORICAL AND CRITICAL ENQUIRY

INTO THE

INTERPRETATION OF THE HEBREW SCRIPTURES,

WITH REMARKS ON

MR. BELLAMY'S NEW TRANSLATION.

BY J. W. WHITTAKER, M.A.

Fellow of St. John's College, Cambridge.